A Faster Exponential Time Algorithm for

Bin Packing with Constant Number of Bins

with the Help of Additive Combinatorics

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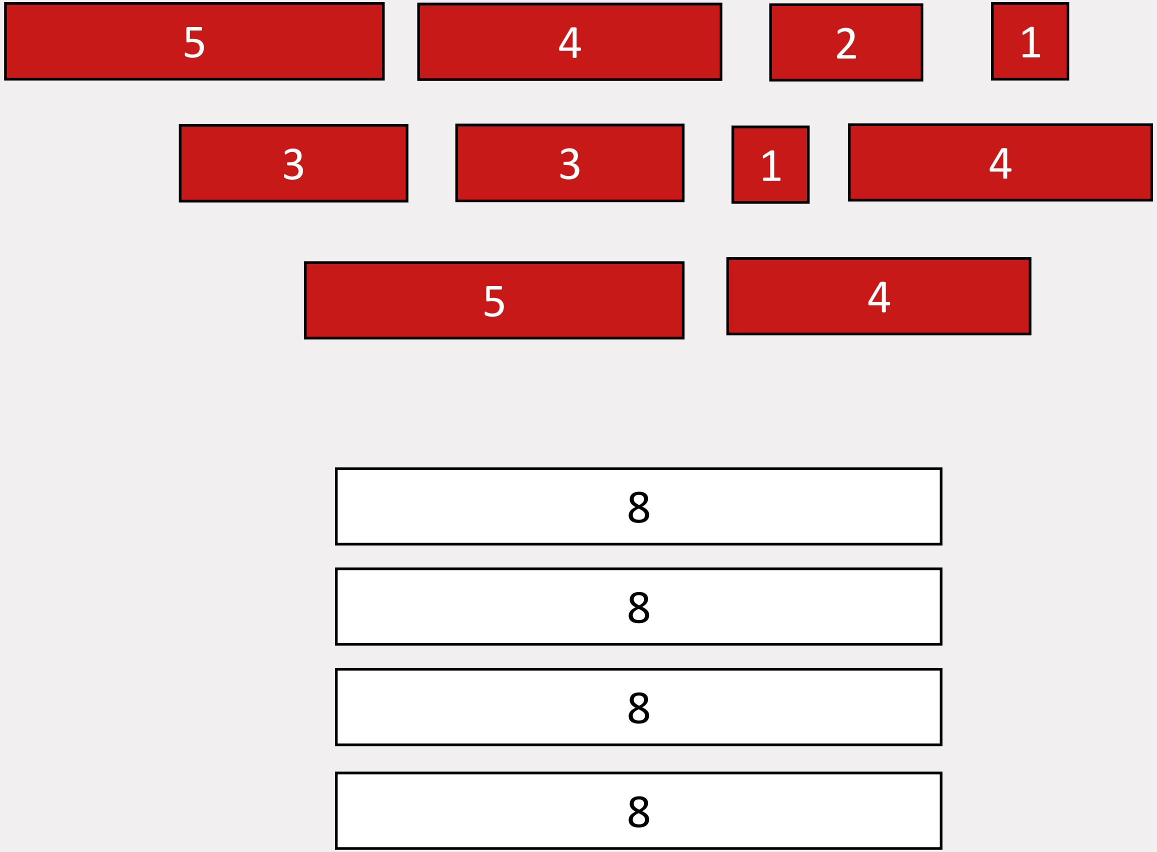
Bin Packing

Given:

- n items
- $w(j)$ weight of item j
  - $w(X) = \sum_{j \in X} w(j)$
- m bins with capacity c

Goal: **distribute items over bins**



Bin Packing

$\alpha(w) = |\{w(X): X \subseteq [n]\}|$
$\mathcal{B}(w) = \{X: w(X) = s\}$

- Algorithm A
  - Dynamic Programming
  - $O(c^m \cdot n \cdot m)$
- Algorithm B
  - Björklund, Husfeldt and Koivisto (SICOMP 2009)
  - $O(2^n \cdot n)$

Key idea:

Algorithm A in $O\left((1.999)^n\right)$ time

If $\alpha(w)$ is small

or

Algorithm B in $O\left((1.999)^n\right)$ time

If $\mathcal{B}(w)$ is small

Open Question

Can do in $O(1.99999^n)$ time?

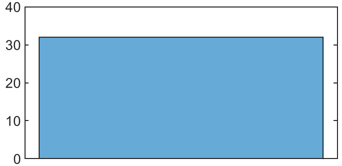
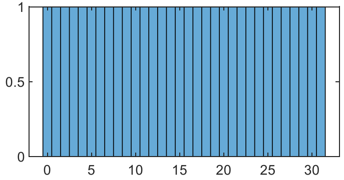
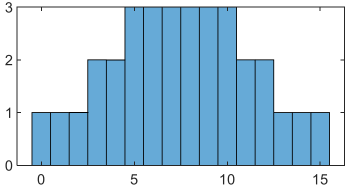
Before our work, only known for $m = 2,3$ (Lente et al.)

Our result:

Can do in $O\left((2 - \varepsilon_m)^n\right)$ time with $\varepsilon_m > 0$ that depends on m.

Parameters $\alpha(w)$ and $|\mathcal{B}(w)|$

$$\alpha(w) = |\{w(X) : X \subseteq [n]\}|$$
$$\mathcal{B}(w) = \{X : w(X) = s\}$$

$w_1, \dots, w_5$	Histogram	$\alpha(w)$	$\max|\mathcal{B}(w)|$
0 0 0 0 0		1	32
1 2 4 8 16		32	1
1 2 3 4 5		16	3

Additive Combinatorics

$$\alpha(w) = |\{w(X): X \subseteq [n]\}|$$
$$\mathcal{B}(w) = \{X: w(X) = s\}$$

If $\alpha(w) \leq 1.99^{n/m}$
Algorithm A runs in time
$O(1.999^n)$

If $|\mathcal{B}(w)| \leq 1.99^n$,
Algorithm B runs in time
$O(1.999^n)$

Because we have table entries
$\mathrm{D}P(j, c_1, \ldots, c_m)$
$\in \{\text{generated weights}\}$
$\in \{1, \ldots, n\}$

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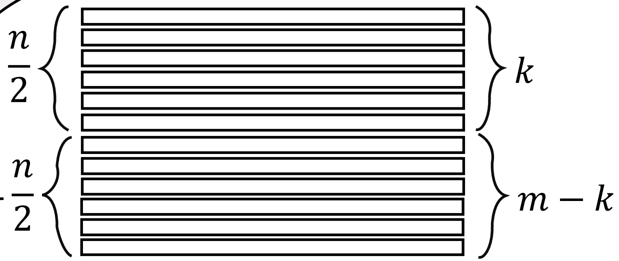

$$\mathcal{B}(w) := \left\{X : |X| = \frac{n}{2}, w(X) = k \cdot c\right\}$$

$\alpha(w) = |\{w(X) : X \subseteq [n]\}|$
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Additive Combinatorics

If $\alpha(w) \leq 1.99^{n/m}$
Algorithm A runs in time
$O(1.999^n)$

If $|\mathcal{B}(w)| \leq 1.99^n$,
Algorithm B runs in time
$O(1.999^n)$

Theorem: For all $\delta > 0$ there exists $\varepsilon > 0$ s.t.

if $|\mathcal{B}(w)| \geq 2^{(1-\varepsilon)n}$, then $\alpha(w) \leq 2^{\delta n}$.

How to prove this?

Take $\delta < \frac{1}{m}$

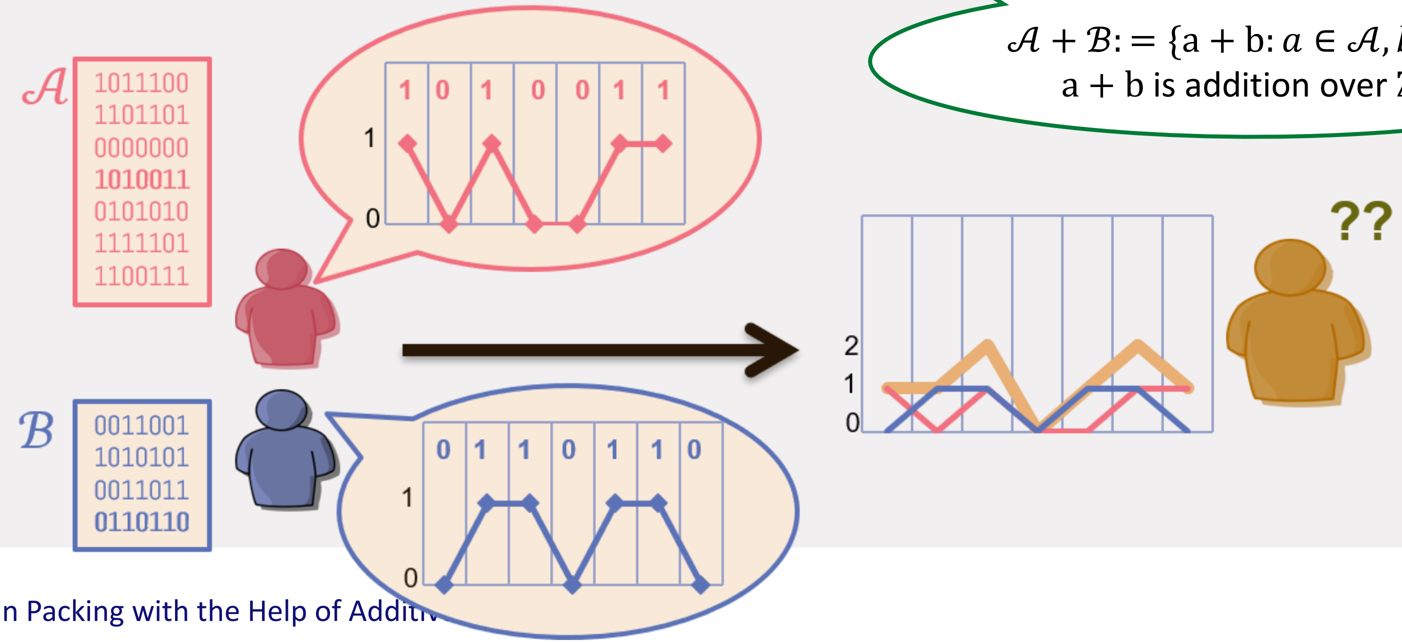
Additive Combinatorics

Def: Uniquely Decodable Pairs (UDCP's)

A UDCP is a pair $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^n$ s.t. $|\mathcal{A} + \mathcal{B}| = |\mathcal{A}||\mathcal{B}|$.

$\mathcal{A} + \mathcal{B} := \{\mathrm{a} + \mathrm{b} : a \in \mathcal{A}, b \in \mathcal{B}\}$

a + b is addition over $\mathbb{Z}^n$.



Additive Combinatorics

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Example 1:

$\mathcal{A} = \{000, 100\}$ $\mathcal{B} = \{000, 001, 010, 011\}$

$\mathcal{A} + \mathcal{B} = \{000, 001, 010, \dots\}$

Example 2:

$\mathcal{A} = \{10, 01\}$ $\mathcal{B} = \{00, 01, 11\}$

$\mathcal{A} + \mathcal{B} = \{10, 11, 21, 01, 02, 12\}$

$$\alpha(w) = |\{w(X): X \subseteq [n]\}|$$
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Additive Combinatorics

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$\mathcal{A} \subseteq \{0,1\}^n$ s.t. all $a \in \mathcal{A}$ **have different weight** $|\mathcal{A}| = \alpha(w)$

$\mathcal{B} \subseteq \{0,1\}^n$ s.t. all $b \in \mathcal{B}$ **have weight s** $|\mathcal{B}| = |\mathcal{B}(w)|$

$\mathcal{A}$ and $\mathcal{B}$ is UDCP:

Let c be received.

$c = a + b$, so $\langle w, c \rangle = \langle w, a \rangle + \overset{s}{\cancel{\langle w, b \rangle}}.$

$\Rightarrow a$ was used!

$b = c - a.$

See w as a vector,

$$w(X) = \langle w, X \rangle = \left\langle \begin{array}{cc} w(1) & 0 \\ w(2) & 1 \\ w(3), & 0 \\ \vdots & \vdots \\ w(n) & 1 \end{array} \right\rangle$$

Additive Combinatorics

Def: Uniquely Decodable Pairs (UDCP's)

A UDCP is a pair $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^n$ s.t. $|\mathcal{A} + \mathcal{B}| = |\mathcal{A}||\mathcal{B}|$.

Observation: *If* $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^n$ *is UDCP,* $|\mathcal{A}| \cdot |\mathcal{B}| \leq 3^n$.

Because: $\mathcal{A} + \mathcal{B} \subseteq \{0,1,2\}^n$

Corollary: $|\mathcal{A}| = \frac{|\mathcal{A}+\mathcal{B}|}{|\mathcal{B}|} \leq \frac{3^n}{|\mathcal{B}|}$

Additive Combinatorics

Def: Uniquely Decodable Pairs (UDCP's)

A UDCP is a pair $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^n$ s.t. $|\mathcal{A}+\mathcal{B}| = |\mathcal{A}||\mathcal{B}|$.

Best known bounds: *If* $\mathcal{A}, \mathcal{B}$ *is UDCP:*

$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^{1.5\,n}$, [Tilborg, 1978]

If $|\mathcal{A}| \geq 2^{(1-\varepsilon)n}$ then $|\mathcal{B}| \leq 2^{\left(0.4228+\sqrt{\varepsilon}\right)n}$. [Austrin et al. 2018]

We need: If $|\mathcal{B}| \geq 2^{(1-\varepsilon)n}$, then $|\mathcal{A}| \leq 2^{\delta n}$.

$$\alpha(w) = |\{w(X): X \subseteq [n]\}|$$
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Additive Combinatorics

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$\mathcal{B} \subseteq \{0,1\}^n$ s.t. all $b \in \mathcal{B}$ **have weight** $\boldsymbol{s}$ $|\mathcal{B}| = |\mathcal{B}(w)|$

$\boldsymbol{\mathcal{A}}$ **and** $\boldsymbol{\mathcal{B}}$ **is UDCP.**

$k \cdot \mathcal{B} = \{b_1 + \cdots + b_k : b_i \in \mathcal{B}\}$

$\langle w, b\rangle = k \cdot s$ for all $b \in k \cdot \mathcal{B}$.

$\boldsymbol{\mathcal{A}}$ **and** $\boldsymbol{k \cdot \mathcal{B}}$ **is `UDCP'!**

Proof:

Let c be received.

$c = a + b$, so $\langle w, c\rangle = \langle w, a\rangle + \overset{\boldsymbol{k \cdot s}}{\cancel{\langle w, b\rangle}}$.

$\Rightarrow a$ was used!

$b = c - a$.

Additive Combinatorics

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$\mathcal{A}$ and $\mathcal{B}$ is UDCP.

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$\langle w, b\rangle = k \cdot s$ for all $b \in k \cdot \mathcal{B}$.
$\mathcal{A}$ and $k \cdot \mathcal{B}$ is `UDCP'!

Assume $\mathcal{B} \approx \{0,1\}^n$

$\subseteq \{0,\dots,k+1\}^n$

$$|\mathcal{A}| = \frac{|\mathcal{A} + k\cdot\mathcal{B}|}{|k\cdot\mathcal{B}|} \approx\leq \frac{(k+2)^n}{(k+1)^n} = \left(1 + \frac{1}{k+1}\right)^n = 2^{\delta_k n}$$

$\approx \{0,\dots,k\}^n$

$\delta_k \to 0$
as $k \to \infty$

Conclusion

Main result:

Bin Packing in $O\left((2 - \varepsilon_m)^n\right)$ time with $\varepsilon_m > 0$ that depends on m.

Key idea:

New result in Littlewood-Offord theory.

Thanks for your attention!

Future Research:

Bin Packing in $O(1.9999^n)$, m not a constant!