Makespan Scheduling of Unit Jobs with Precedence Constraints in $O(1.995^n)$ time

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The results presented were obtained during the trimester on Discrete Optimization at Hausdorff Research Institute for Mathematics (HIM) in Bonn, Germany.

Department of Mathematics and Computer Science, Eindhoven University of Technology

$$P|prec, p_j = 1|C_{\max}$$

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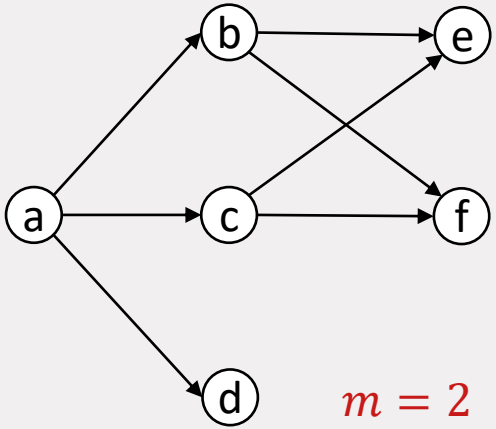

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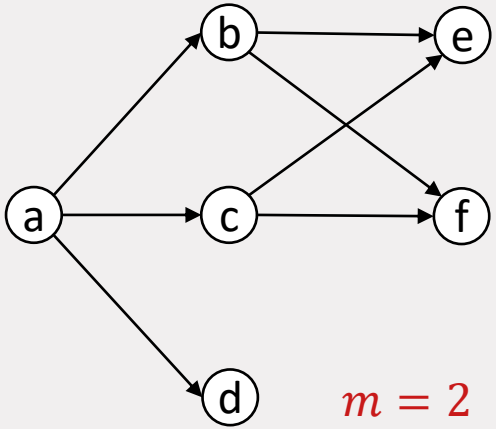

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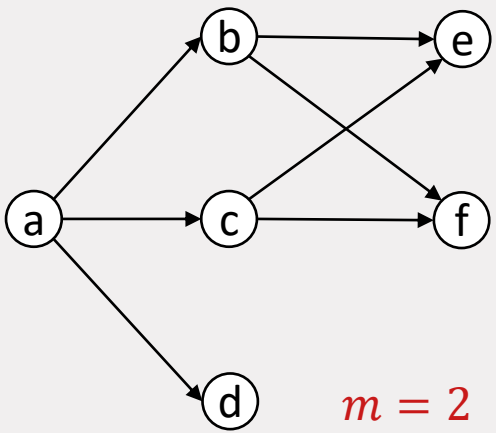

time →

	1	2	3	4
1	a	b	d	f
2		c	e	

Literature overview $P|prec, p_j = 1|C_{\max}$

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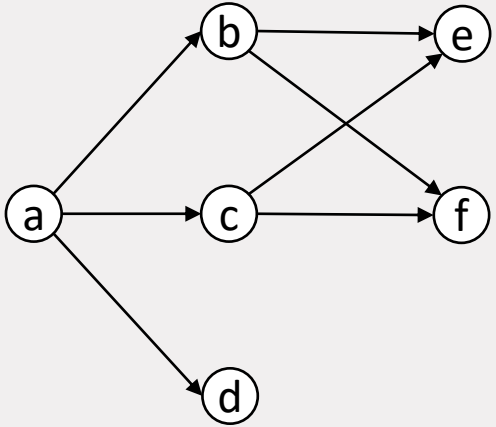
- ???? for $m \geq 3$ **constant** **OPEN**[3]

[3]Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

Definitions

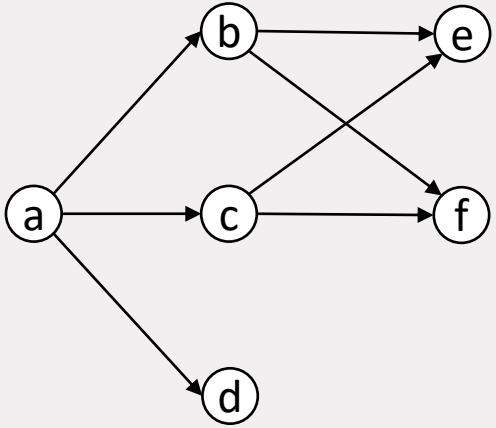


Precendence Constraints Graph G

Definitions

$G \Rightarrow$ partial order:

- $i \prec j$ if $(i, j) \in G$



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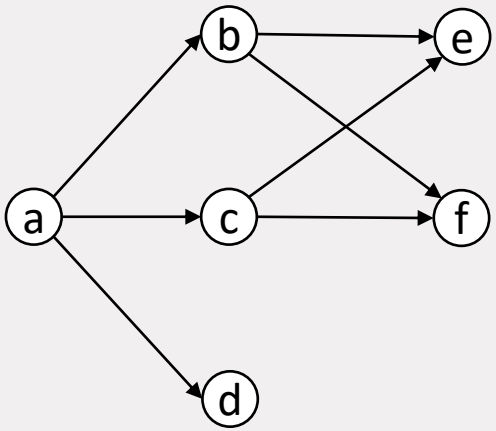

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<u>Definition:</u> Let A be a set of jobs.

$pred[A] \quad = \{x \mid \exists\, a \in A \; s.t. \; x \preccurlyeq a\}$

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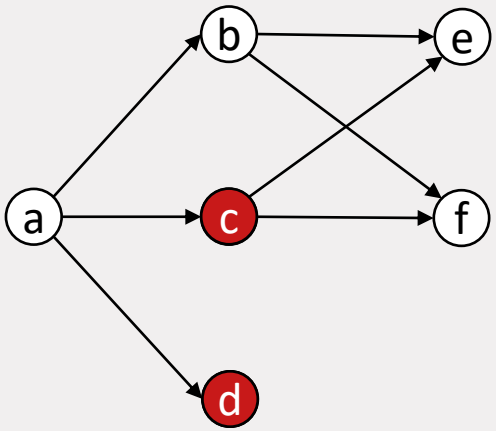

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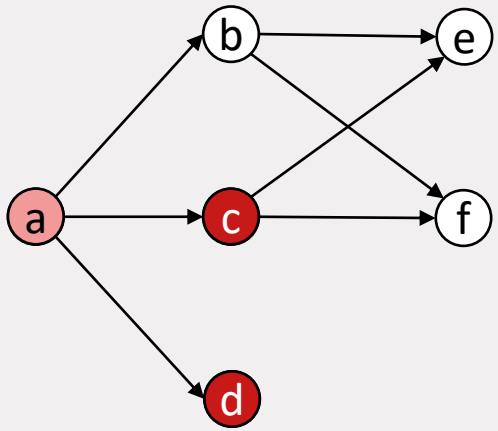

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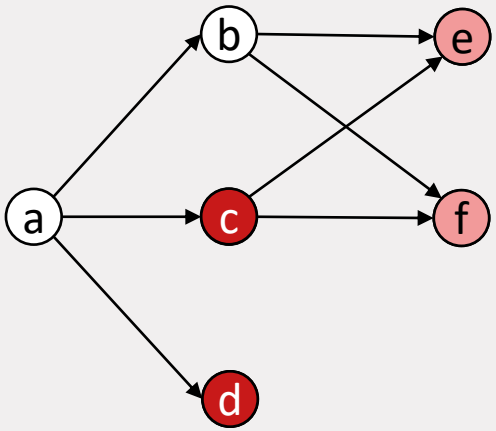

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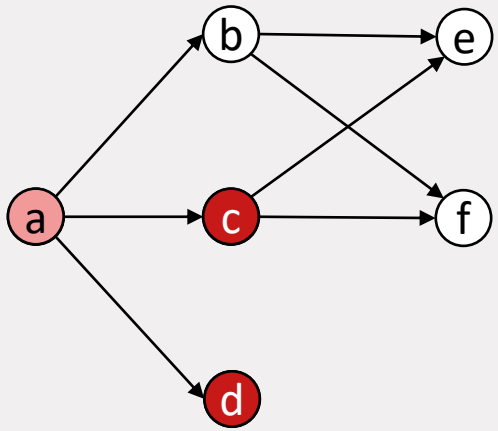

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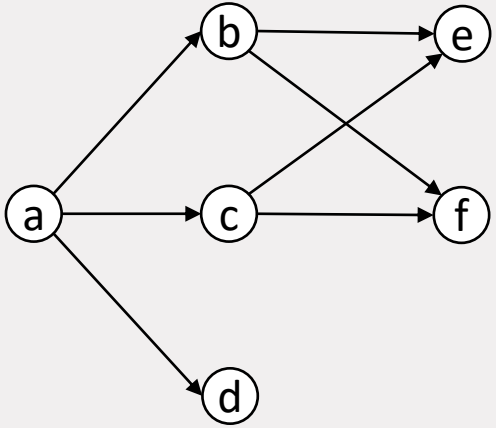

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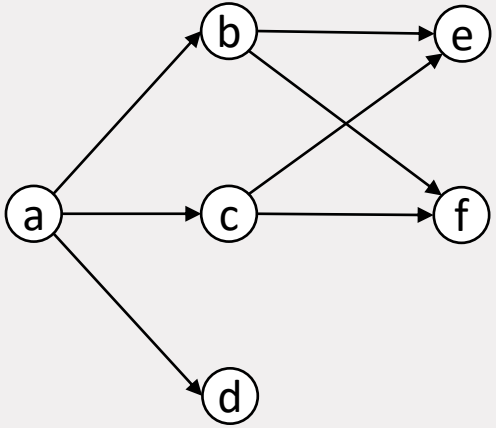
Def: An *antichain* is a set A whose elements are pairwise incomparable.



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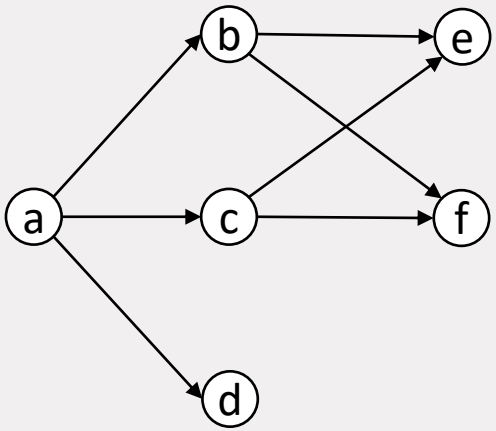

Precendence Constraints Graph G

Ex. of antichains in G

- ✓ $\{b, c, d\}$
- ✓ $\{b, c\}$
- ✓ $\{d, f\}$

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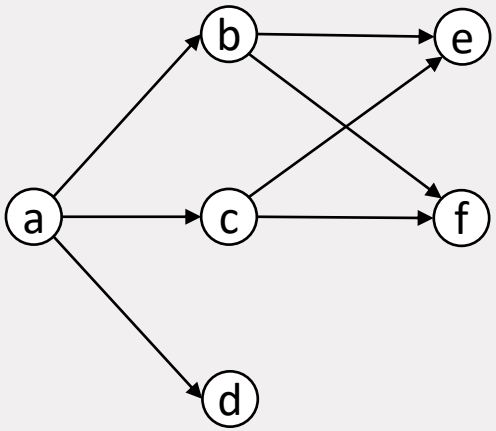

Precendence Constraints Graph G

There is a one-to-one relation between:

A = antichain of graph G

$pred[A]$

$sinks(X)$

X = possible set of jobs to schedule

DP Algorithm:

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$\Rightarrow poly(n) \cdot 2^n \cdot \binom{n}{m}$ time algorithm

Before our work:
No $poly(n) \cdot 2^n$ time algorithm known!

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Our result:

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Proof: Combination of three algorithms

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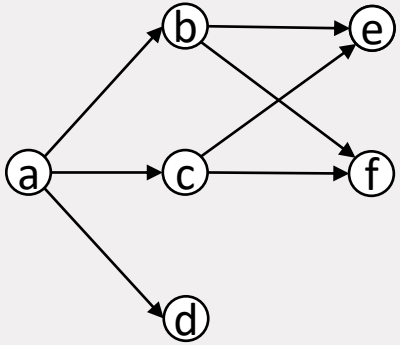

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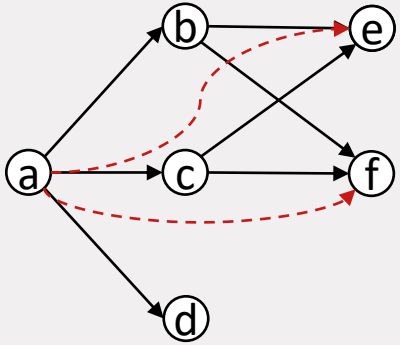
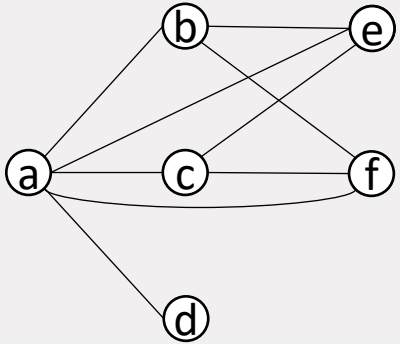


G^{comp}



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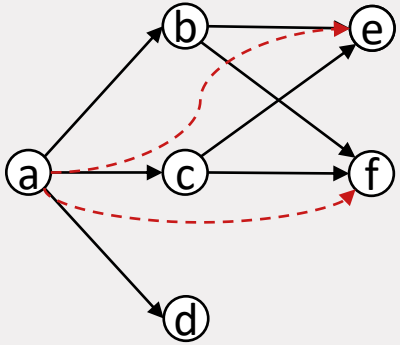

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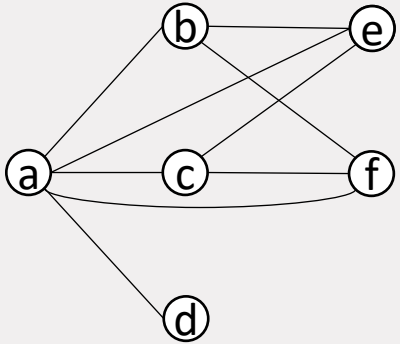

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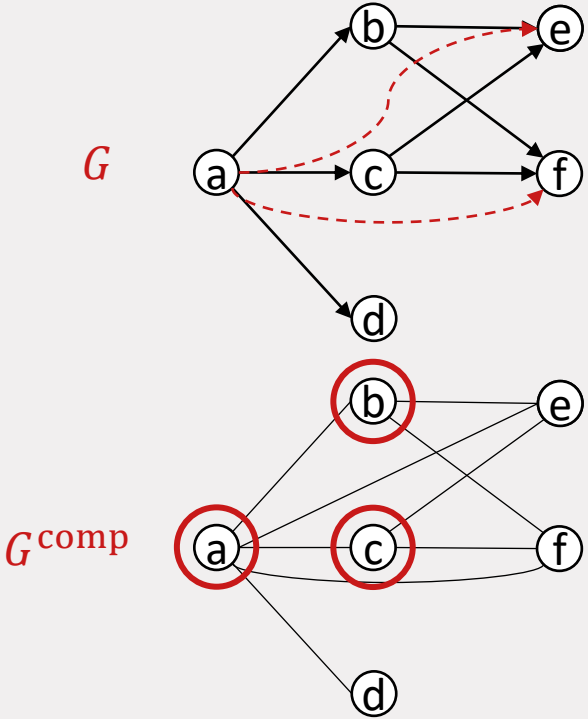

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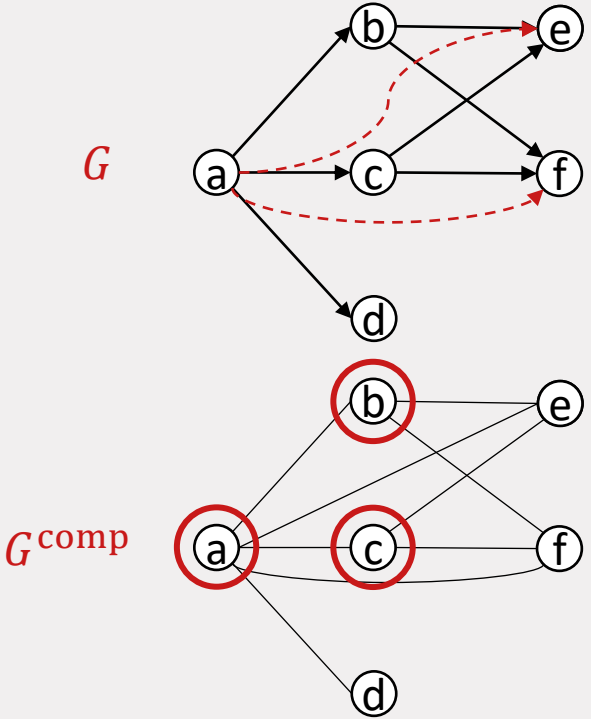
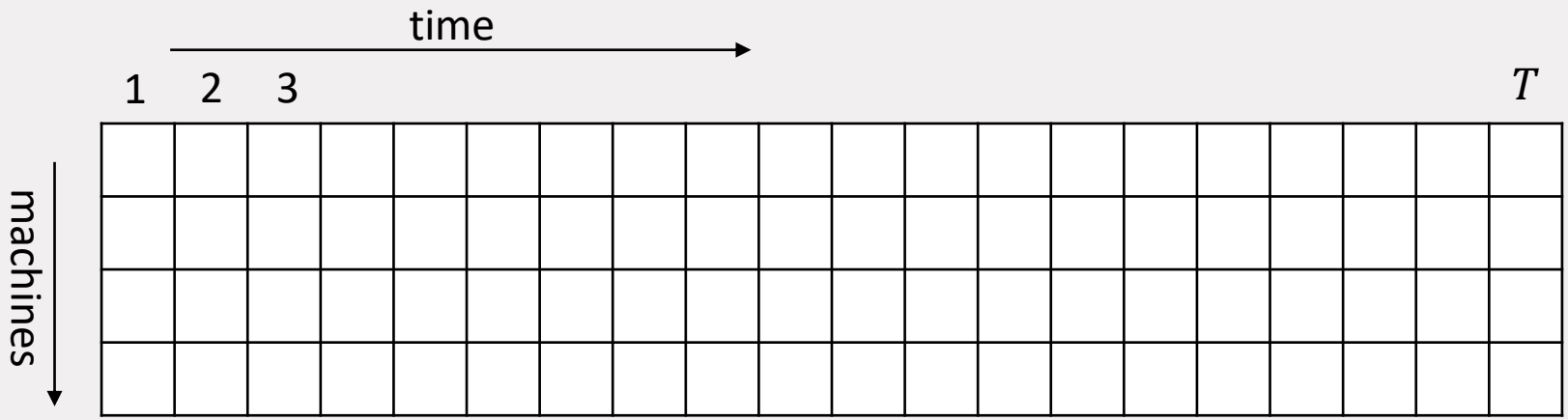
Claim: $V \setminus C$ is an antichain.



Sink-Adjusted Schedule



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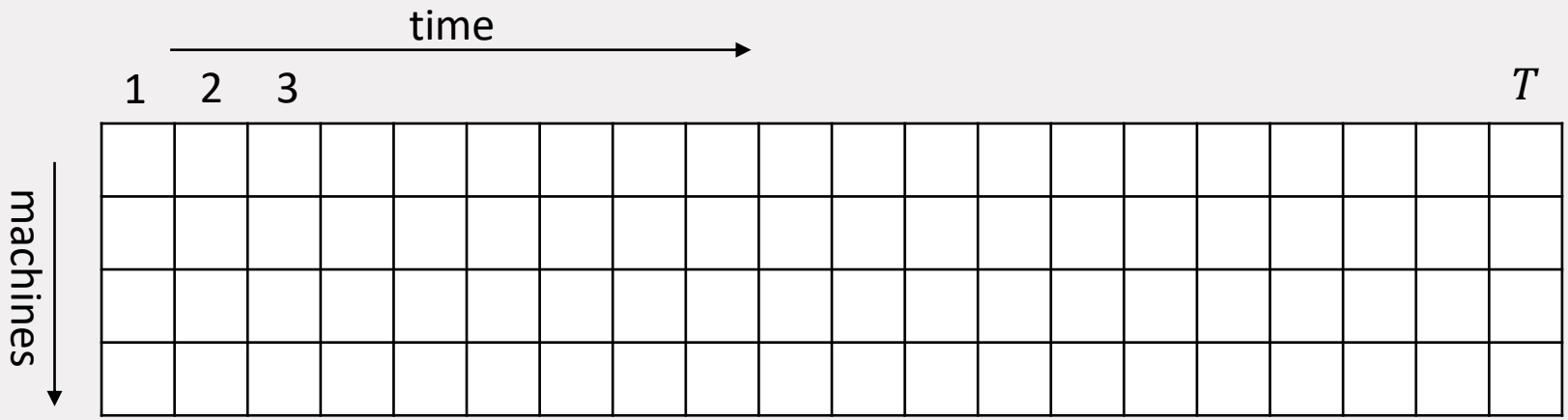

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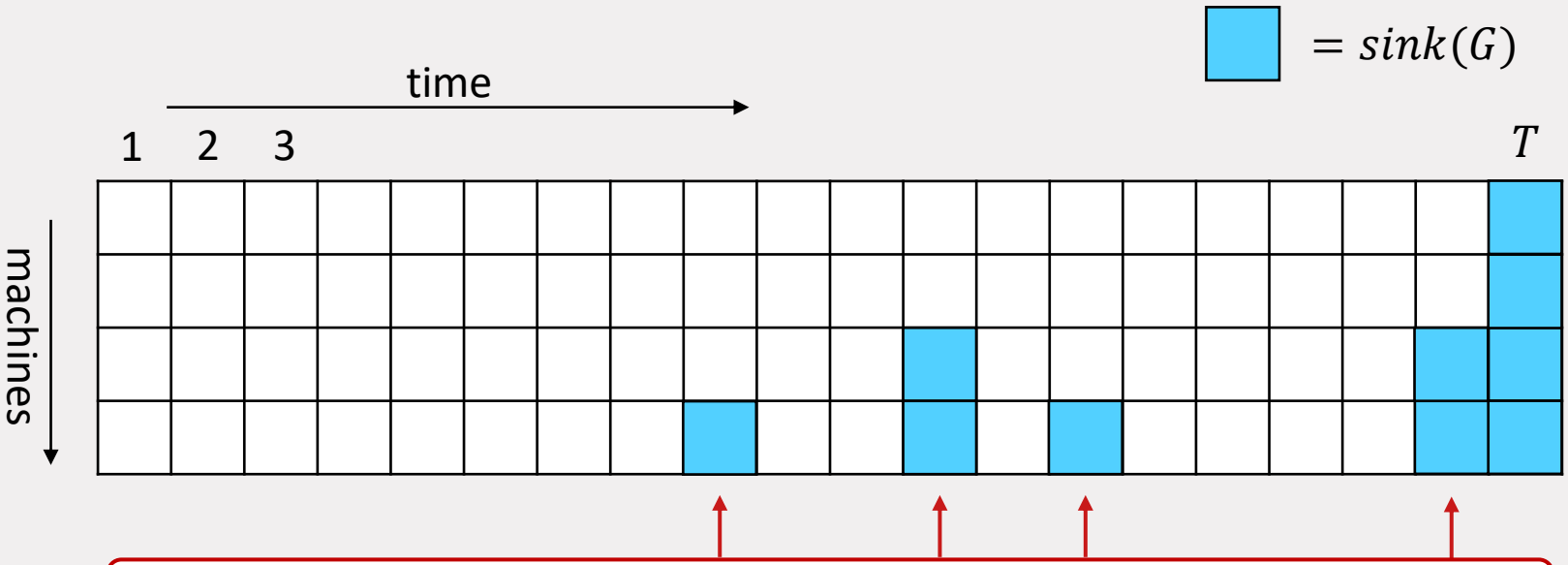

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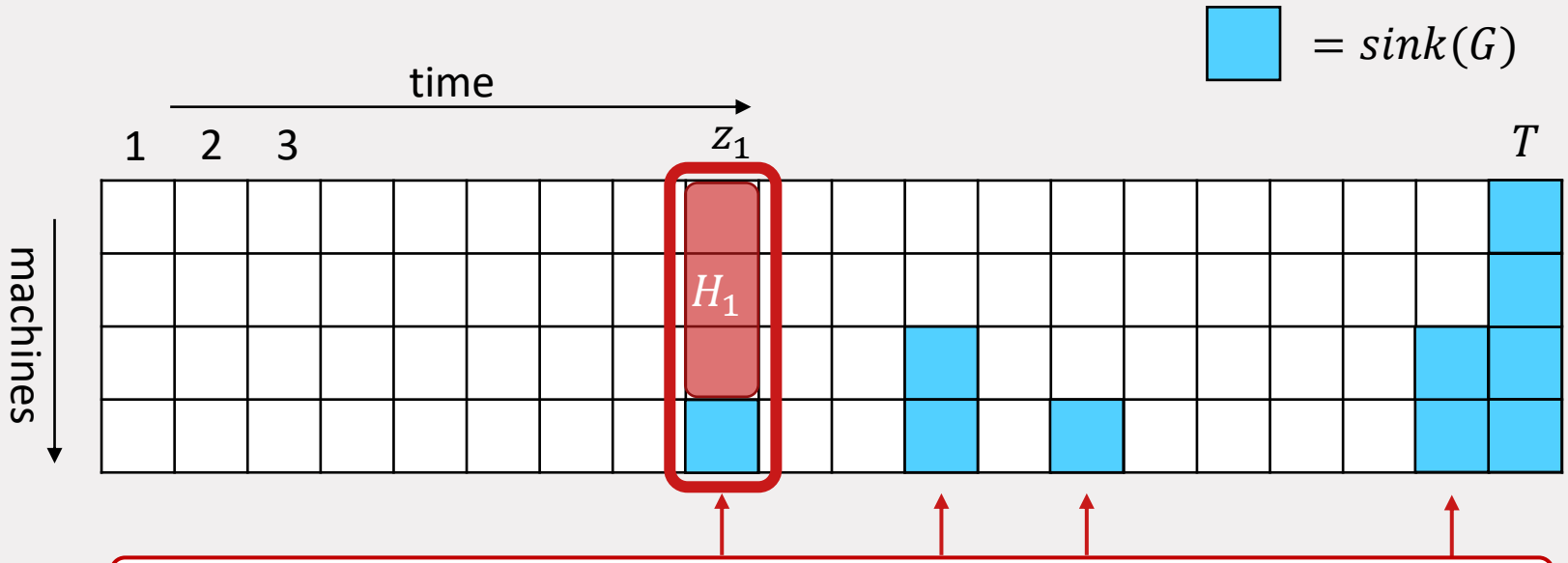

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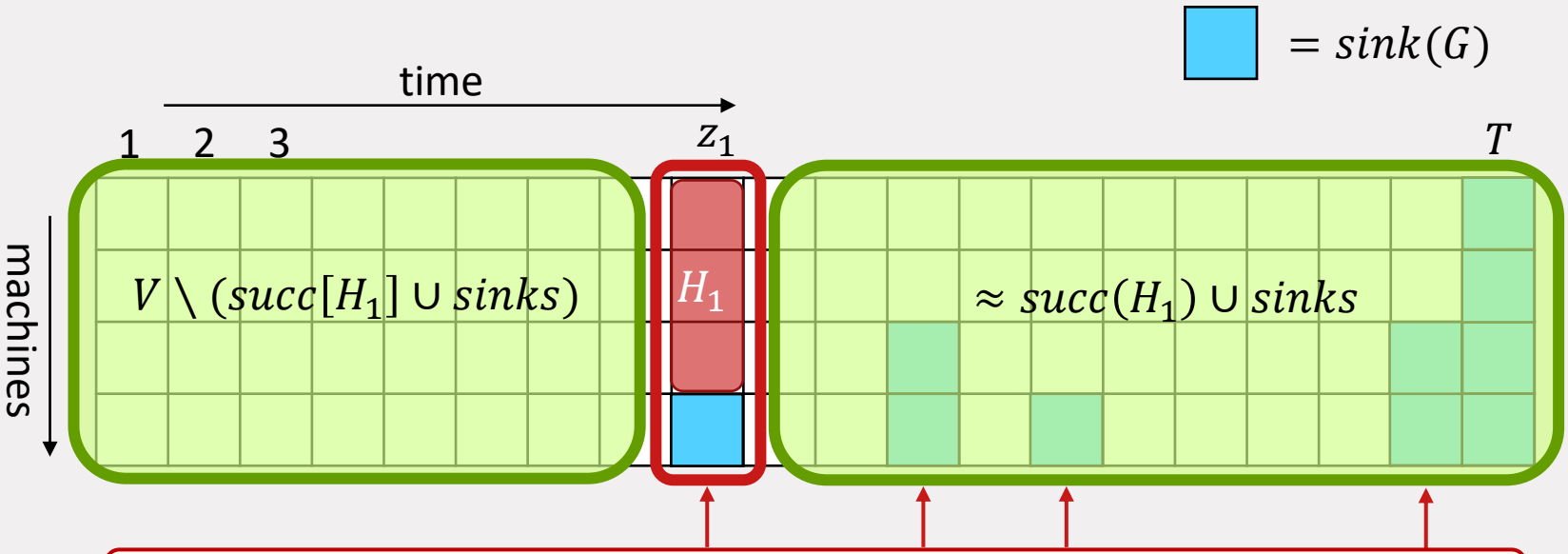

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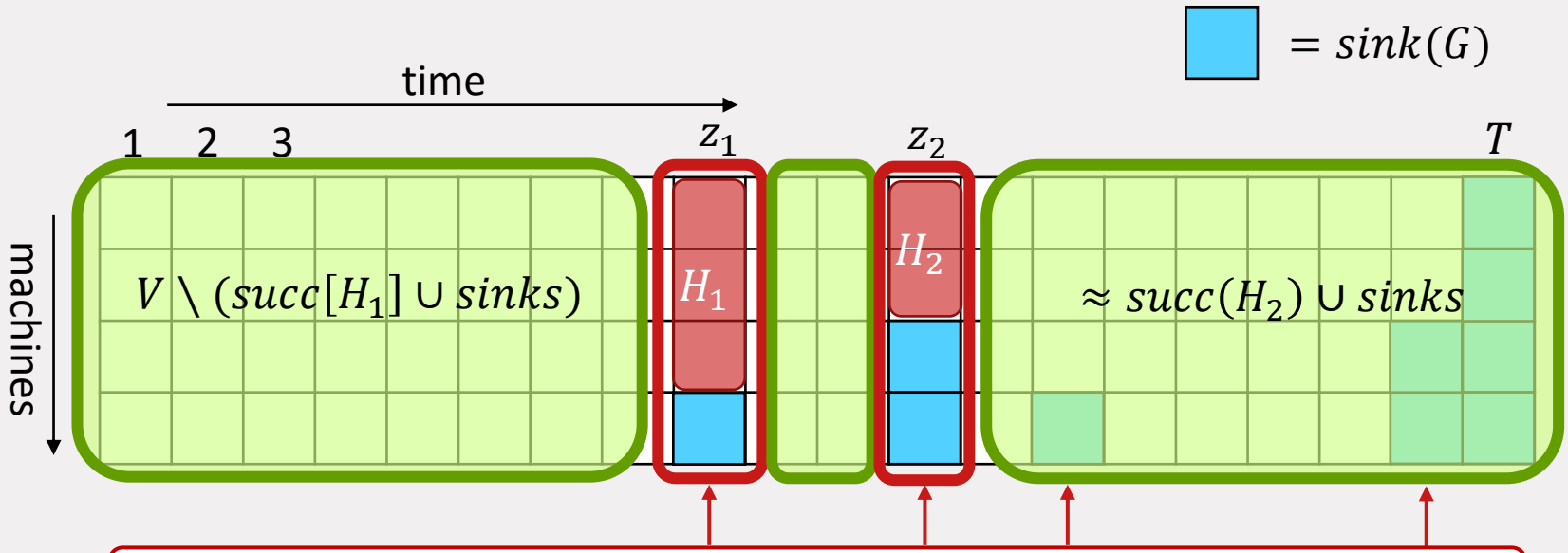

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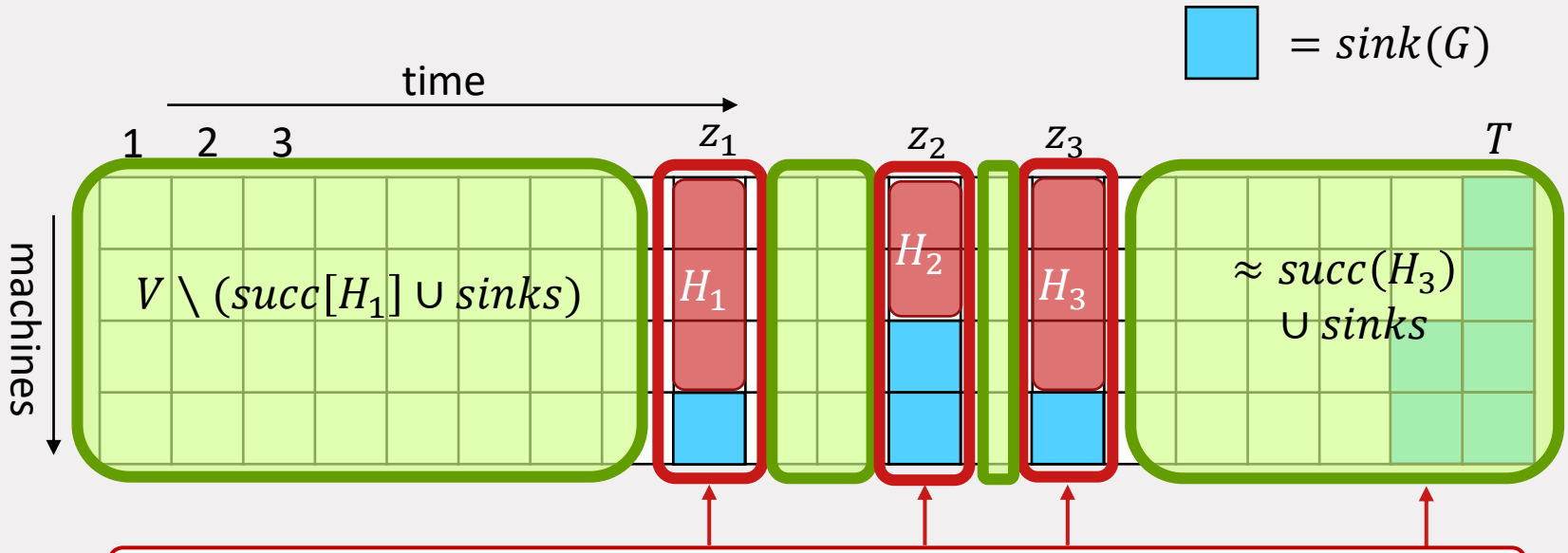

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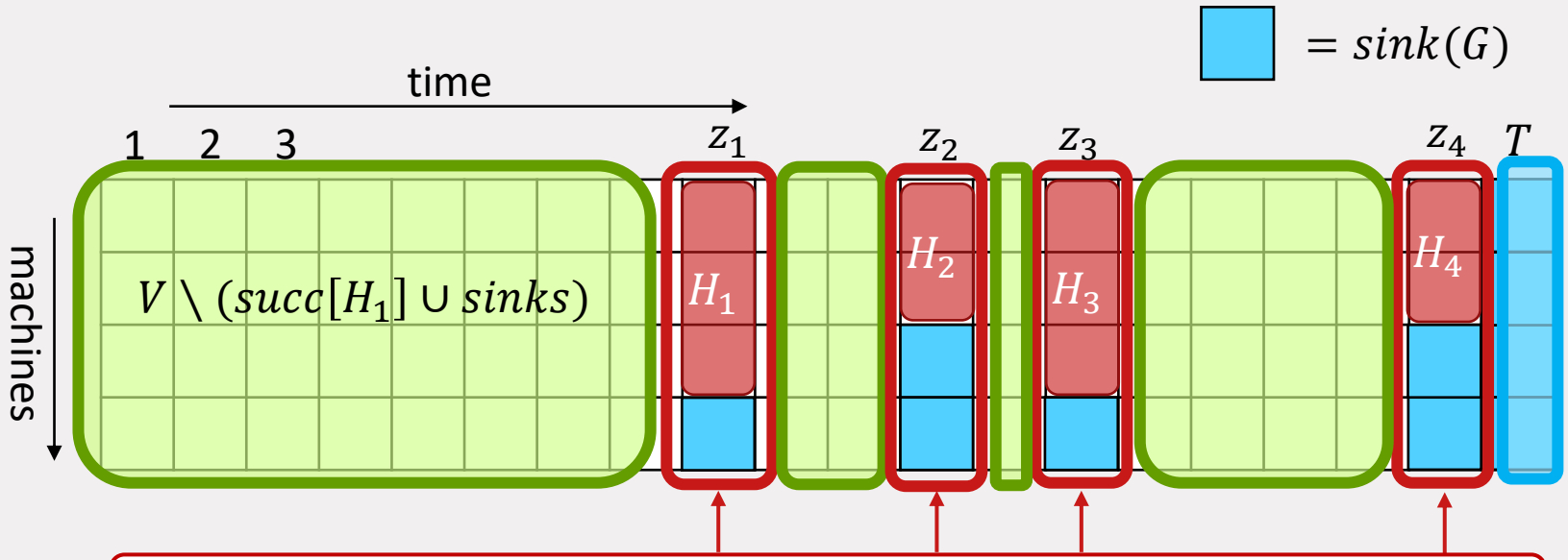

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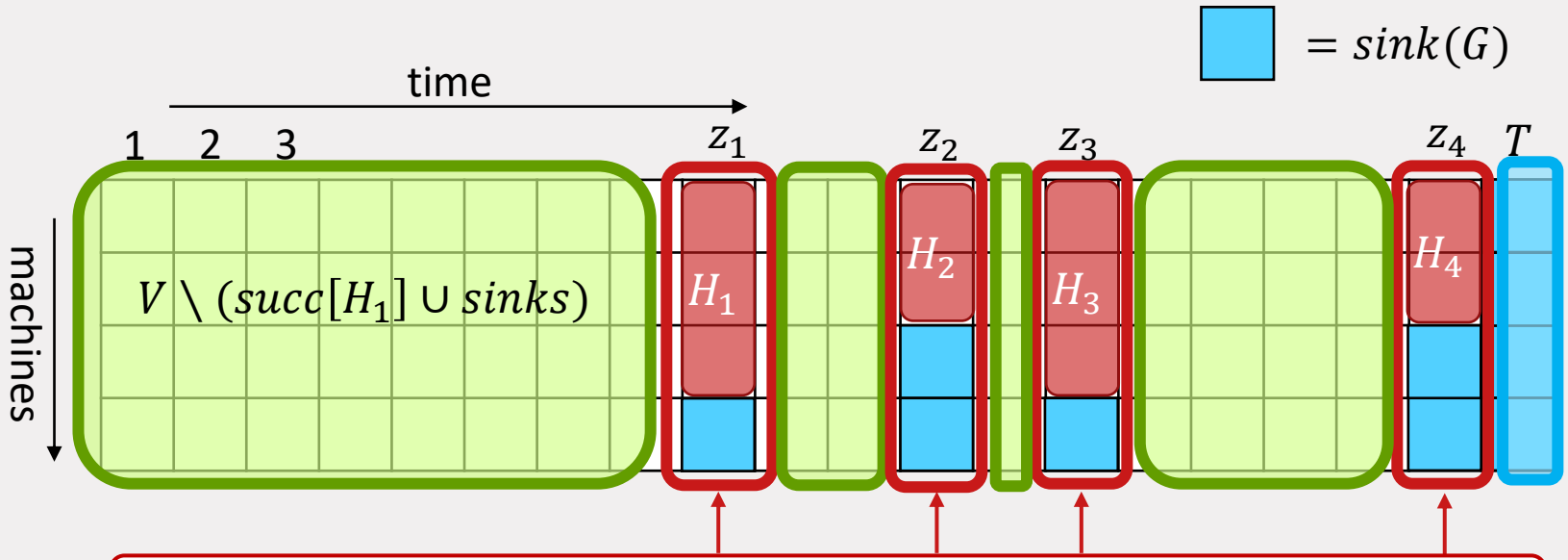
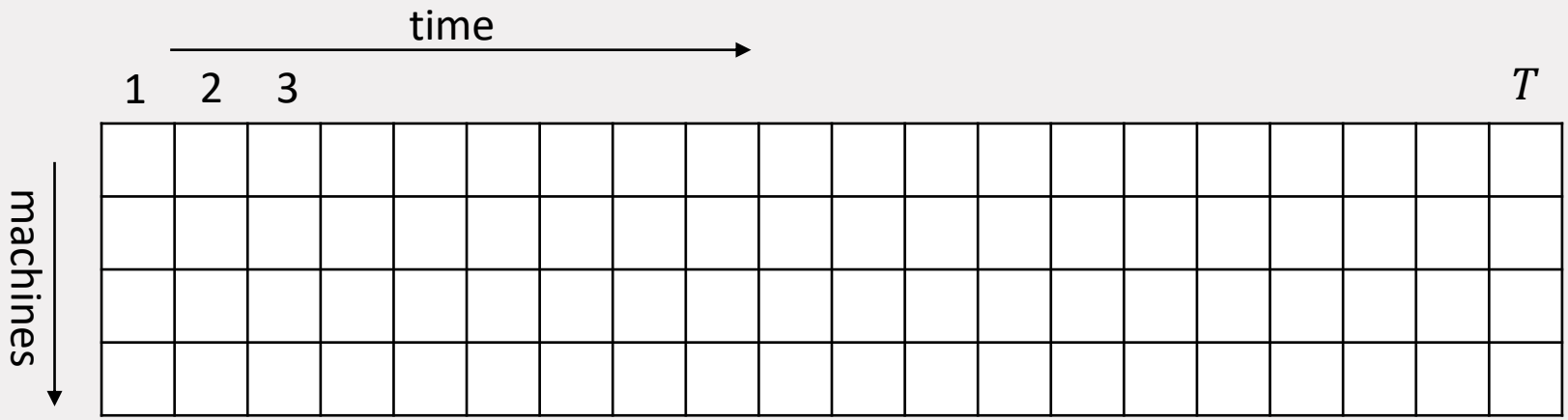


$z \in [1, T]$ is a ***sink moment*** if there are both sinks and non-sinks at time z.

No sinks in

Middle-Adjusted Schedule

Assumption: $n = m \cdot T$



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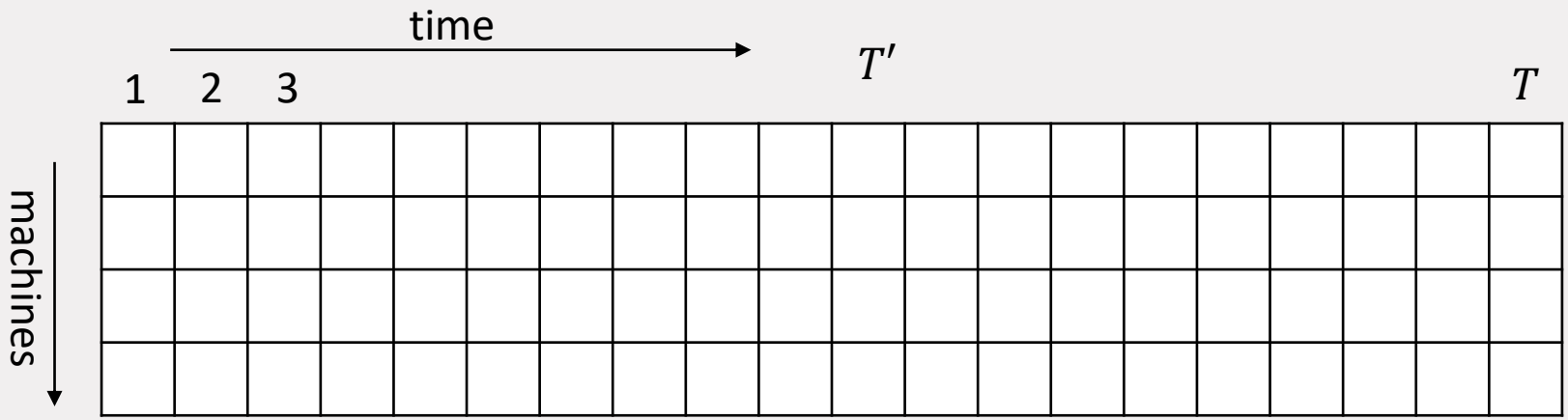

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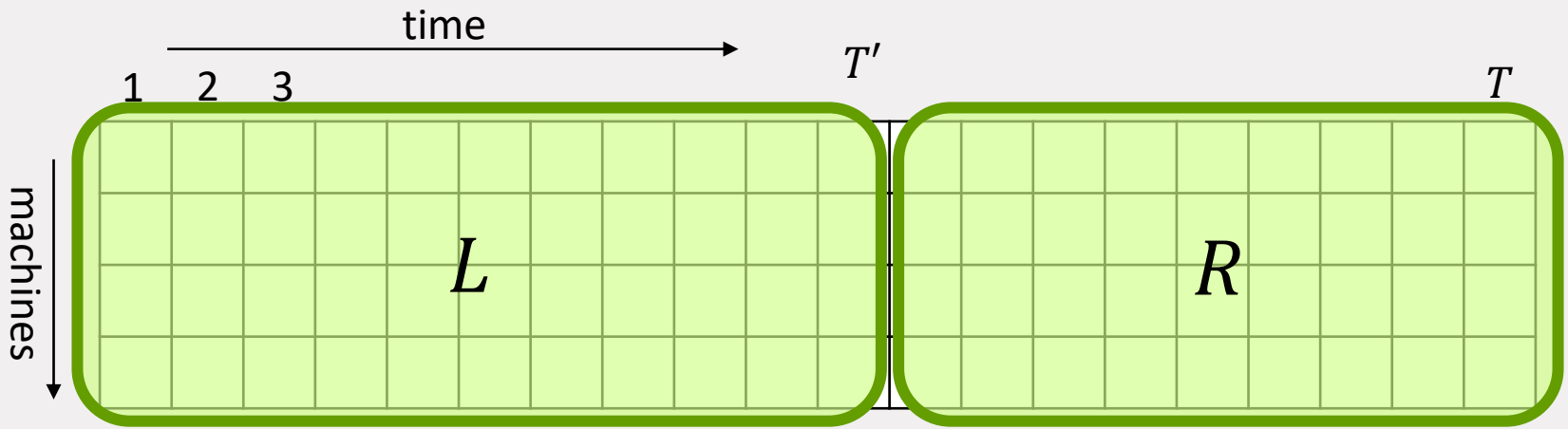

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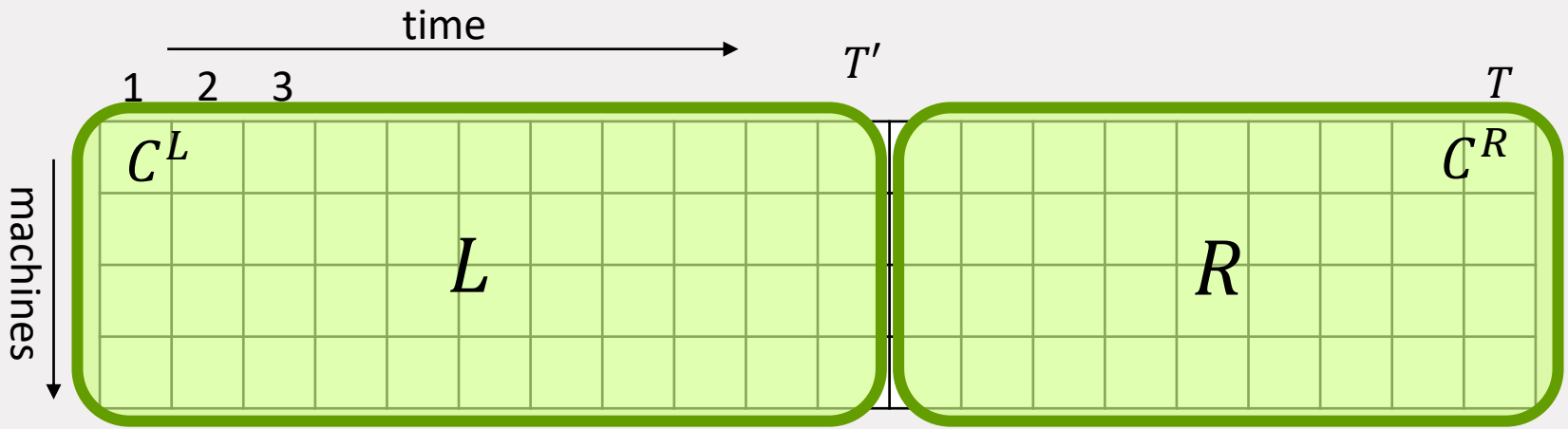

$$|C^L| \approx \frac{|C|}{2} \qquad |C^R| \approx \frac{|C|}{2}$$

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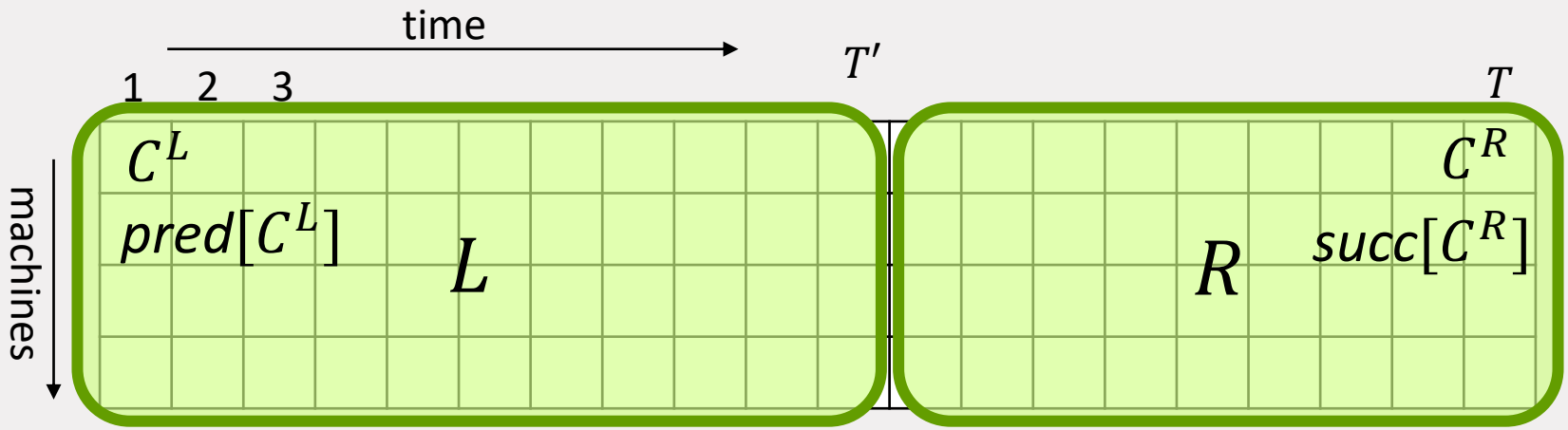

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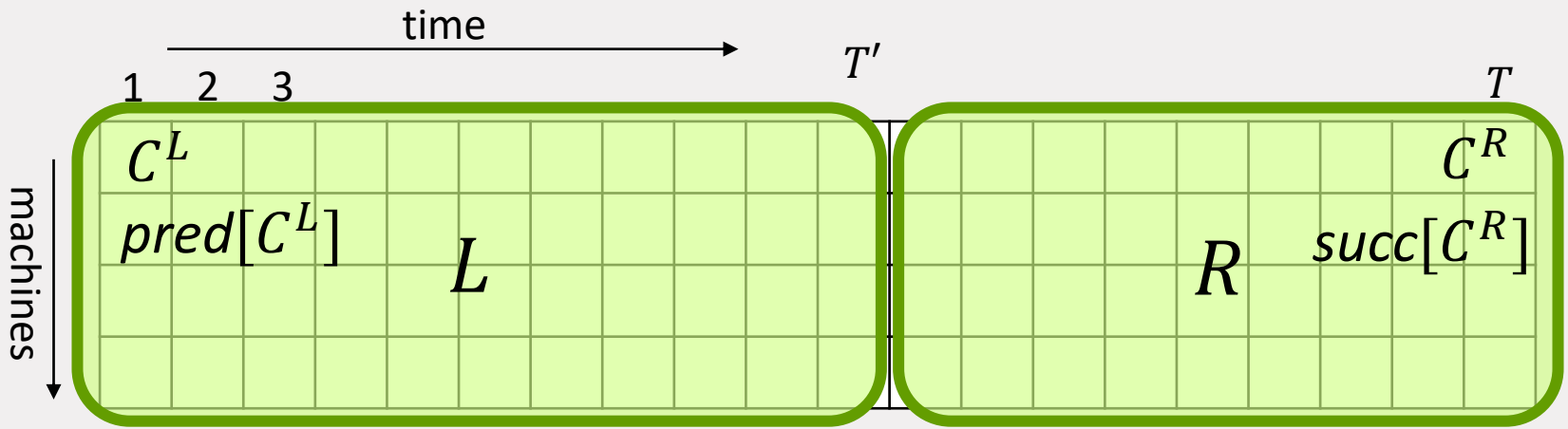

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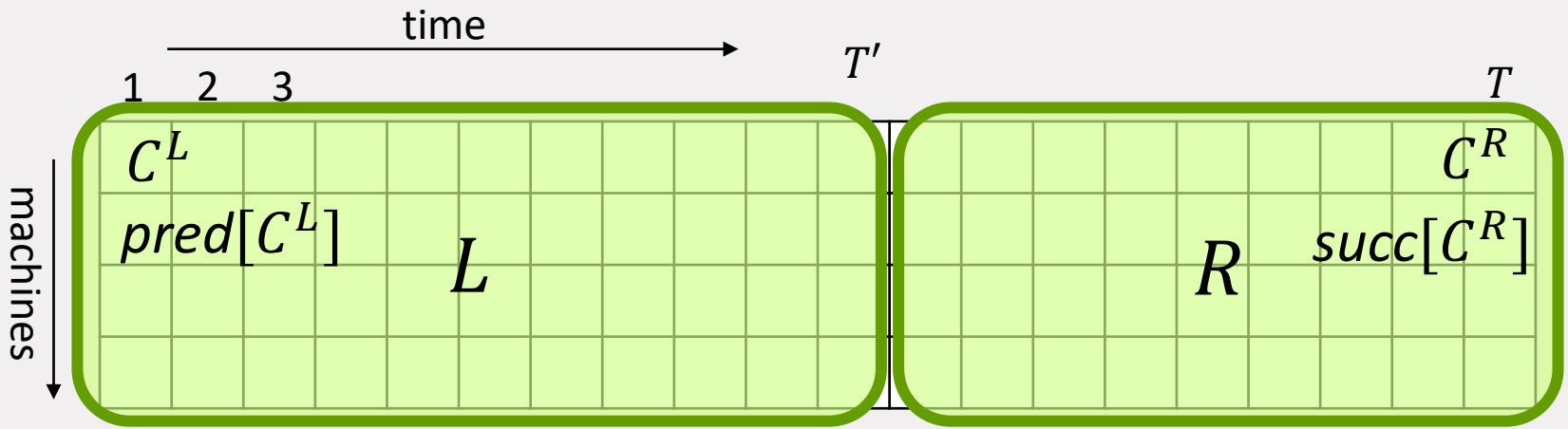

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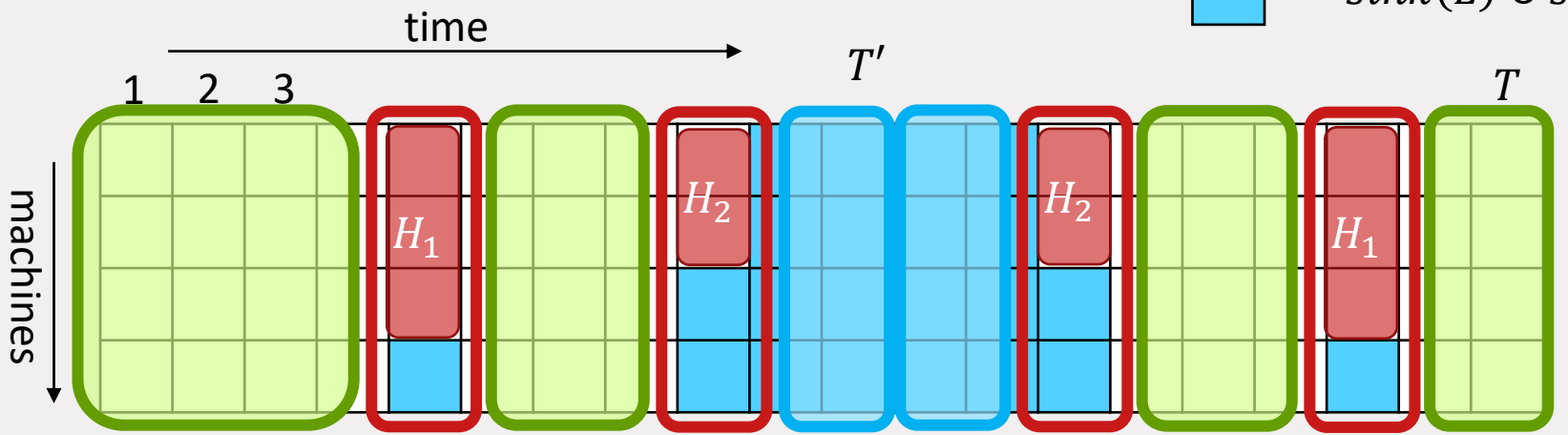
- U are sources in R

Algorithm

Assumption: $n = m \cdot T$

$= sink(L) \cup sink(R)$



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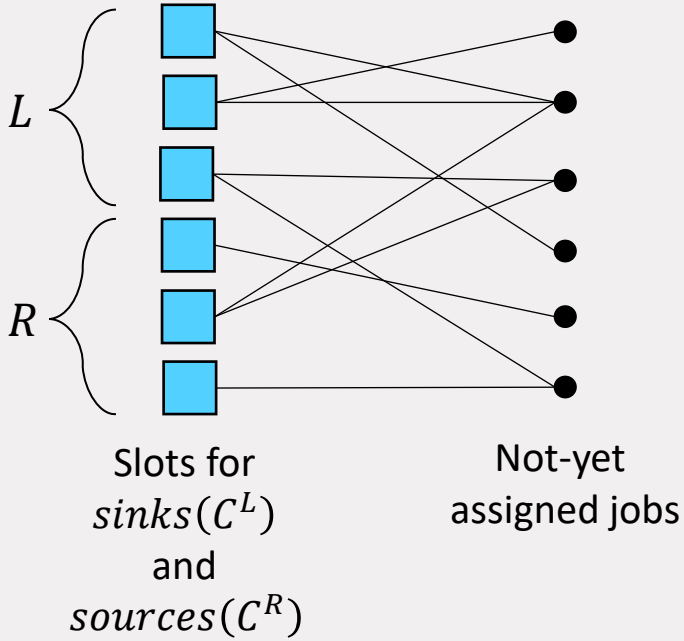

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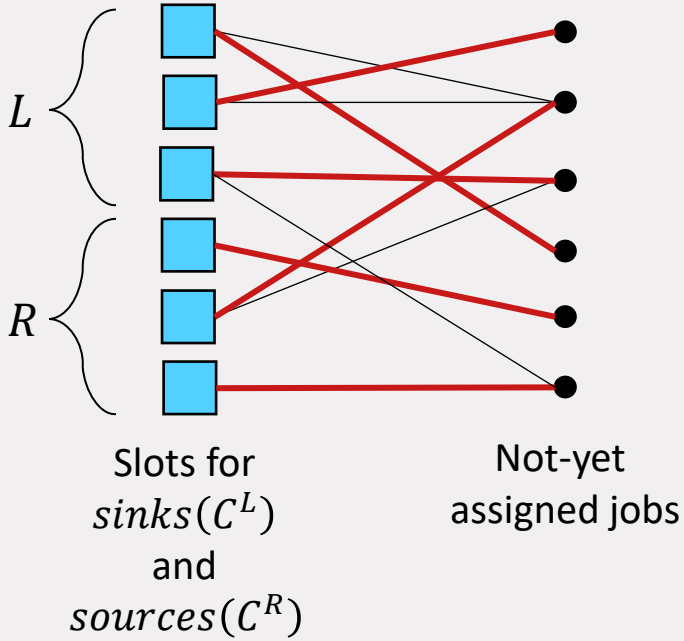
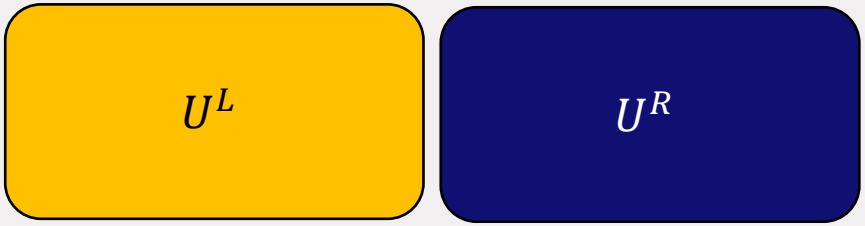


Find partition of U:



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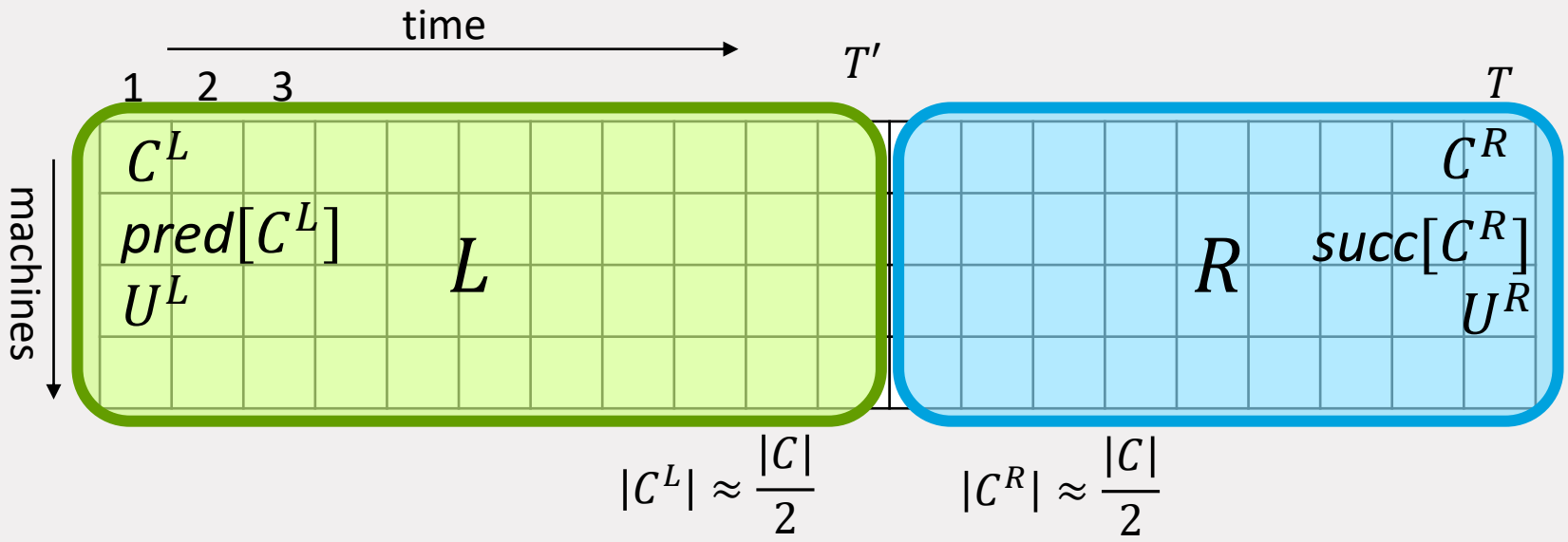

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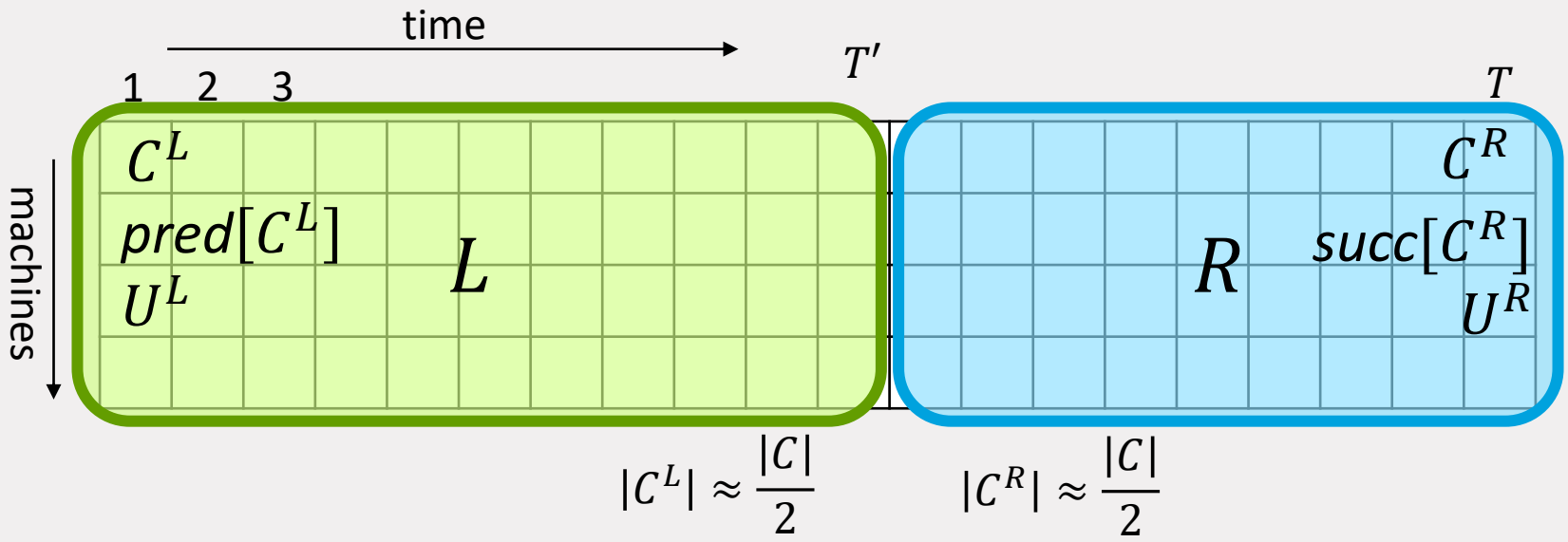

Guesses: $13^{|C|} \Rightarrow O(poly(n) \cdot 169^{|C|})$ time.

Conclusion

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Main result:

$P|prec, p_j = 1|C_{\max}$ in $O(1.995^n)$ time.

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Key idea's:

- Tradeoff between $\#AC's$ and $|C|$.
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Future Research:

$P3|prec, p_j = 1|C_{\max}$ in $2^{o(n)}$ time?

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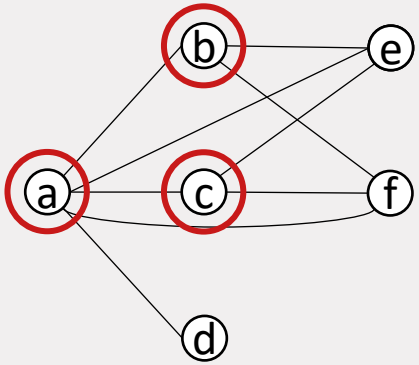
Thanks for your attention!

Size of C vs #AC's

Claim: $|C| \geq n/7.5 \;\Rightarrow\; \#AC's \leq 1.94^n$

G^{comp}



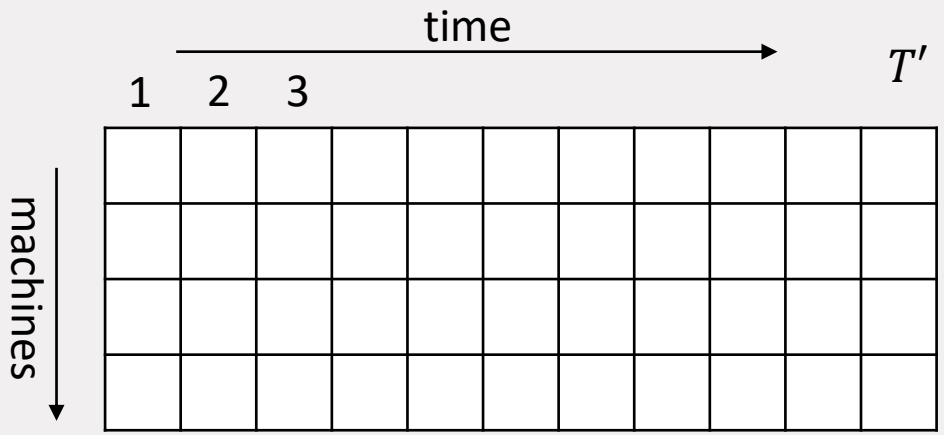
Proof. $|C| \geq n/7.5 \Rightarrow$ largest AC $\leq n - \frac{n}{7.5} = \alpha n$ with $\alpha = \left(1 - \frac{1}{7.5}\right)$.

Dilworth's Theorem: $\exists$ chains $C_1, \ldots, C_{\alpha n}$ that partition G.

$$\#\text{AC's} = \prod_{i=1}^{\alpha n} (|C_i| + 1) \leq \left(\frac{n}{\alpha n} + 1\right)^{\alpha n} = 1.945^n$$

Algorithm

Assumption: $n = m \cdot T$



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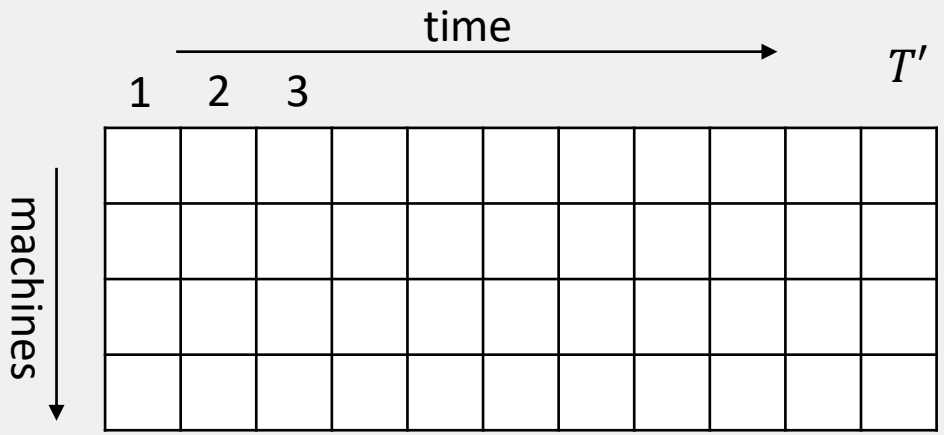

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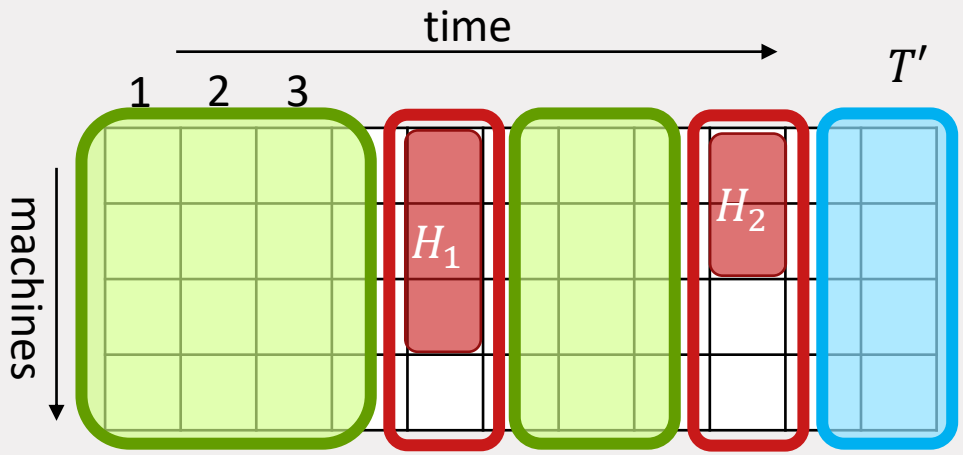

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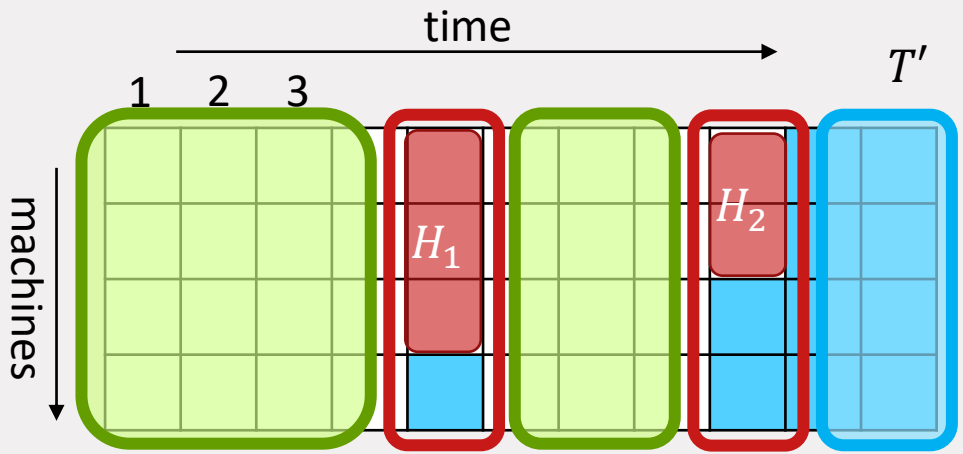

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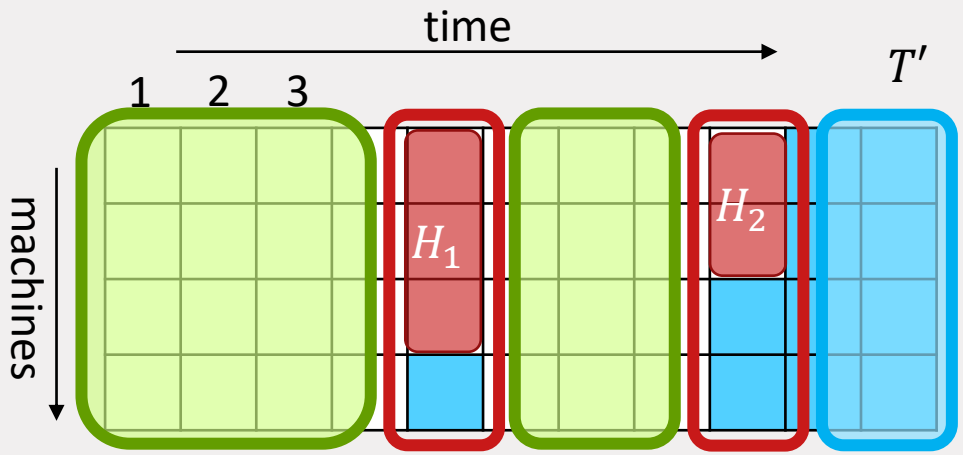

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Assume that $H_1, H_2, \ldots \subseteq C^L$

- Guess jobs in H from C^L
- Reconstruct $H_1, H_2, \ldots$
- Reconstruct
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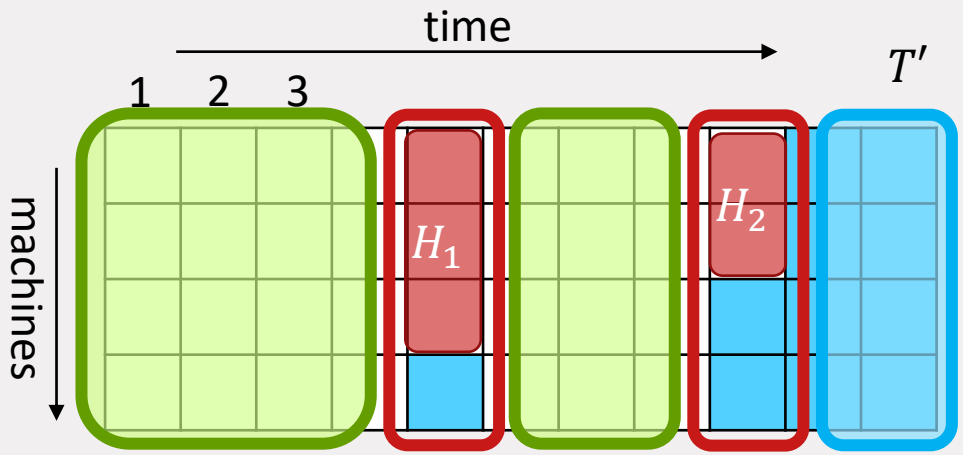

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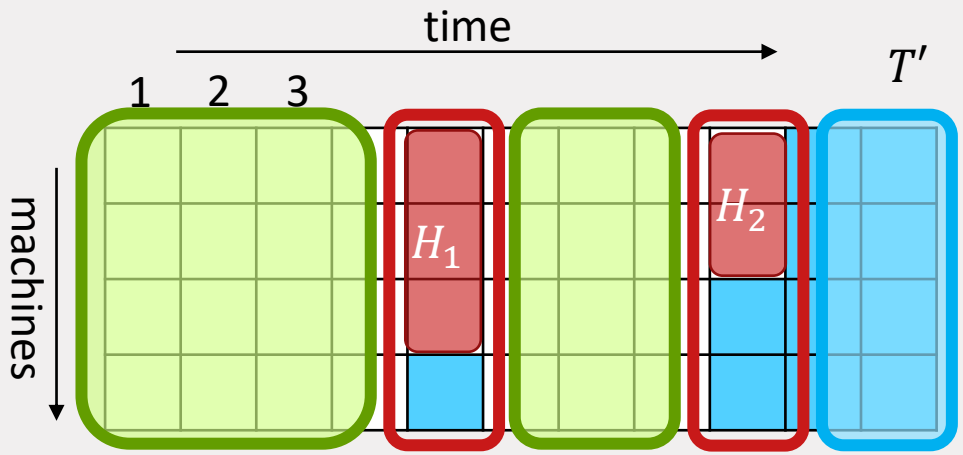

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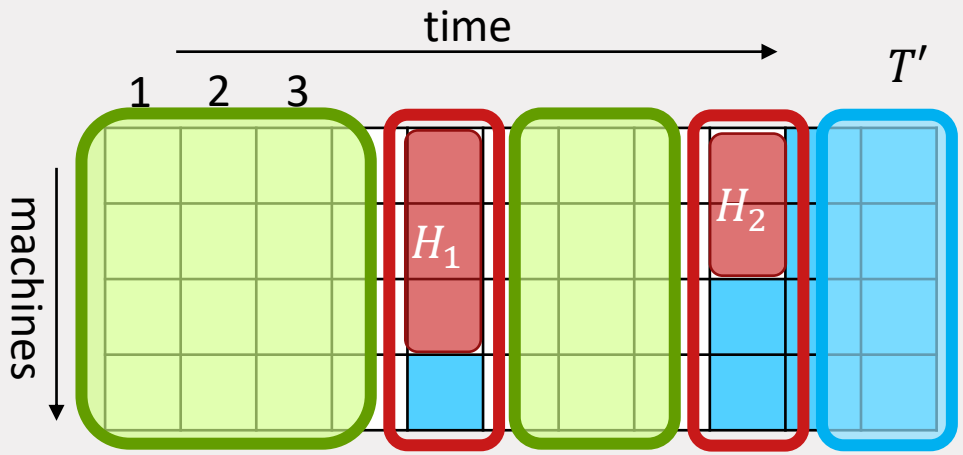

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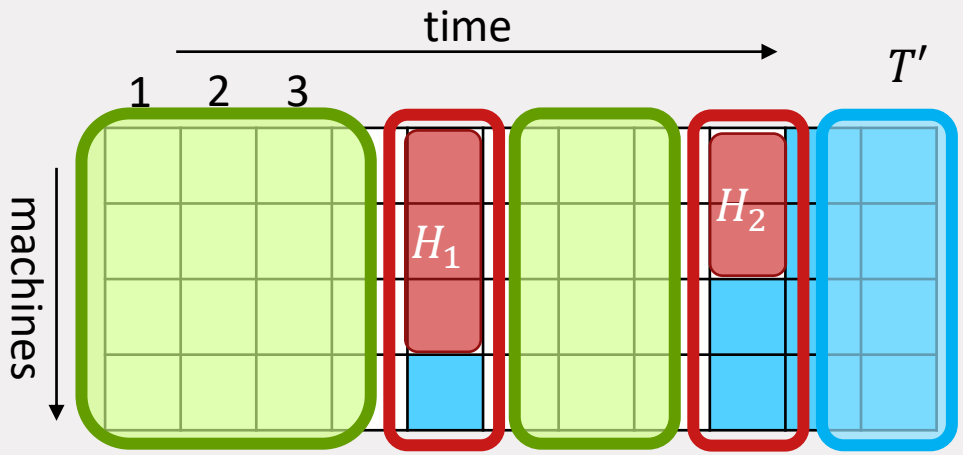

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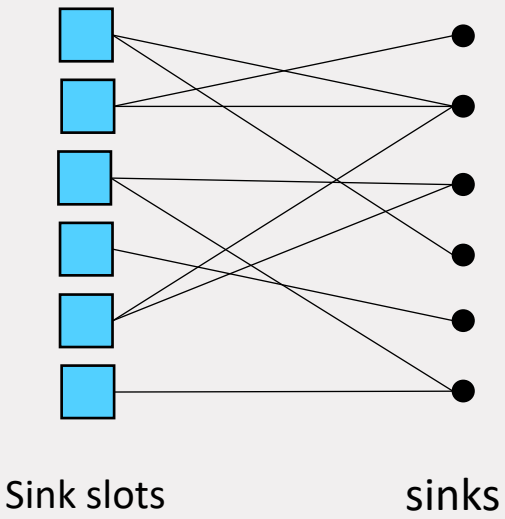
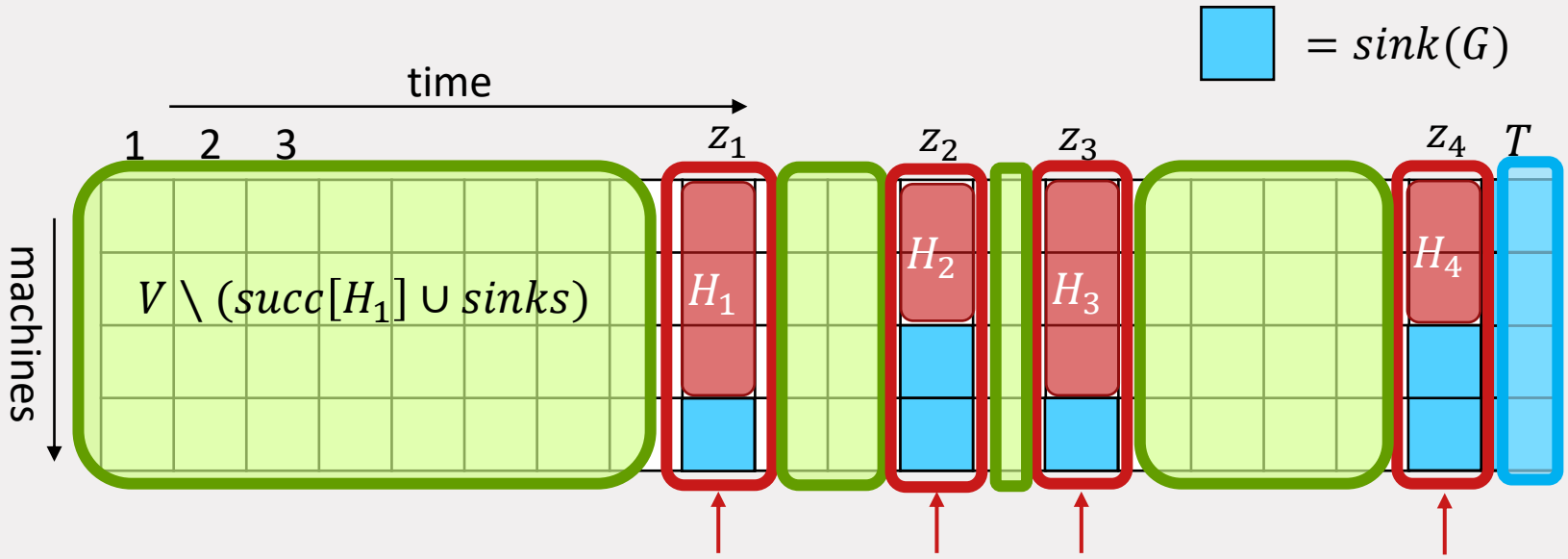


Sink-Adjusted Schedule

Assumption: $n = m \cdot T$



Claim: if we know $\cup_i H_i$, we can derive $H_1, H_2, \ldots, H_l$

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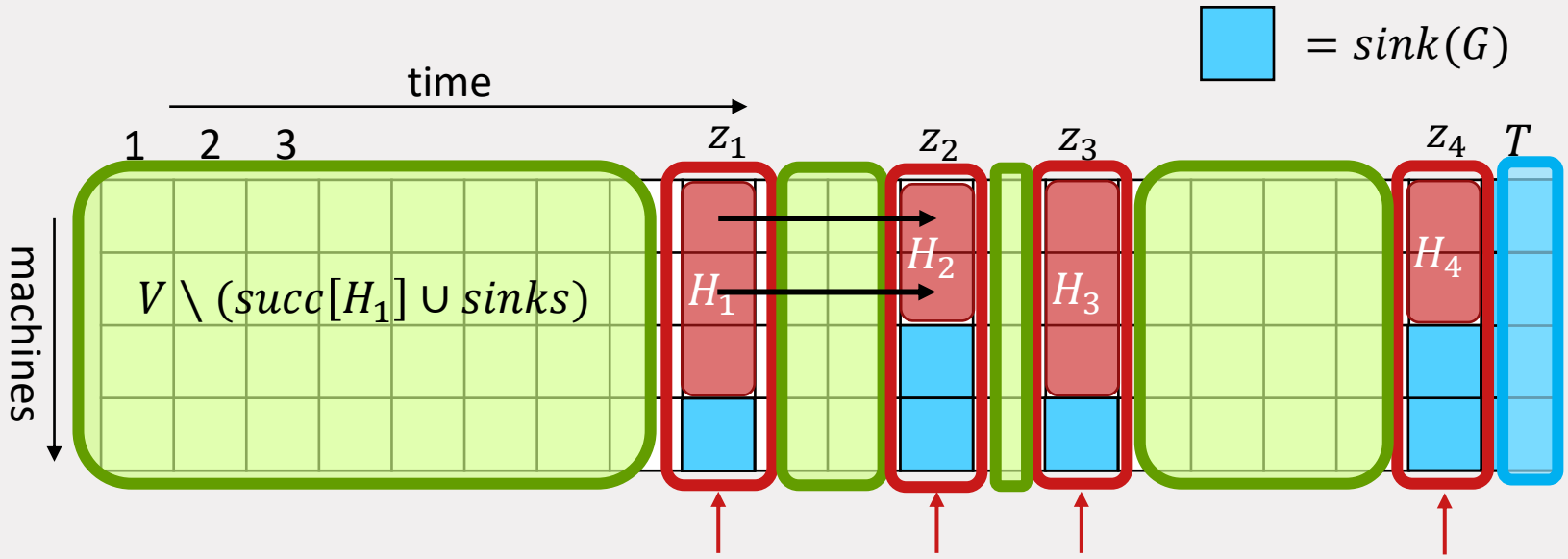

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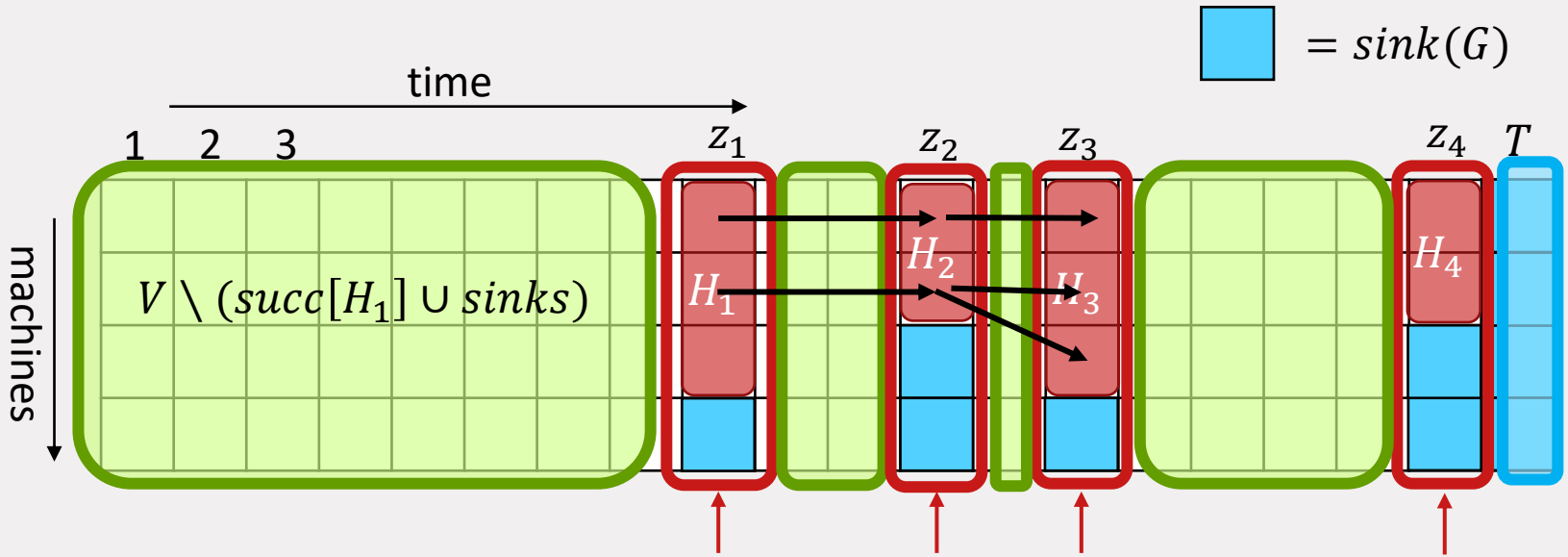

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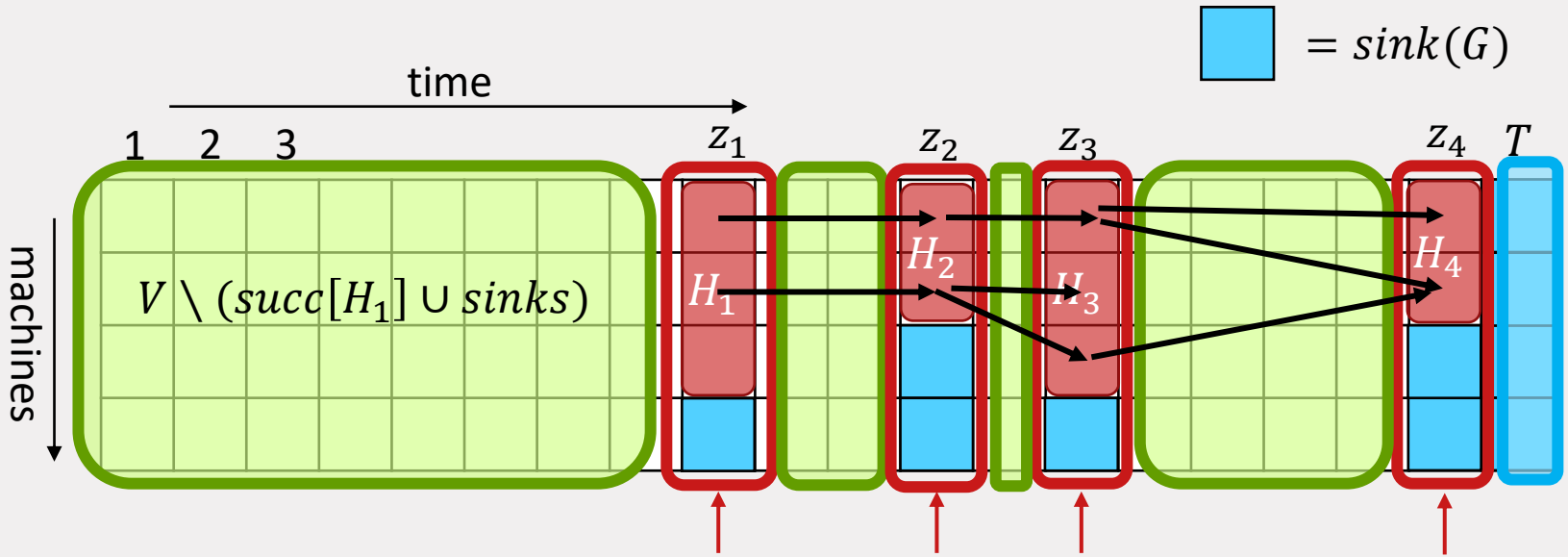

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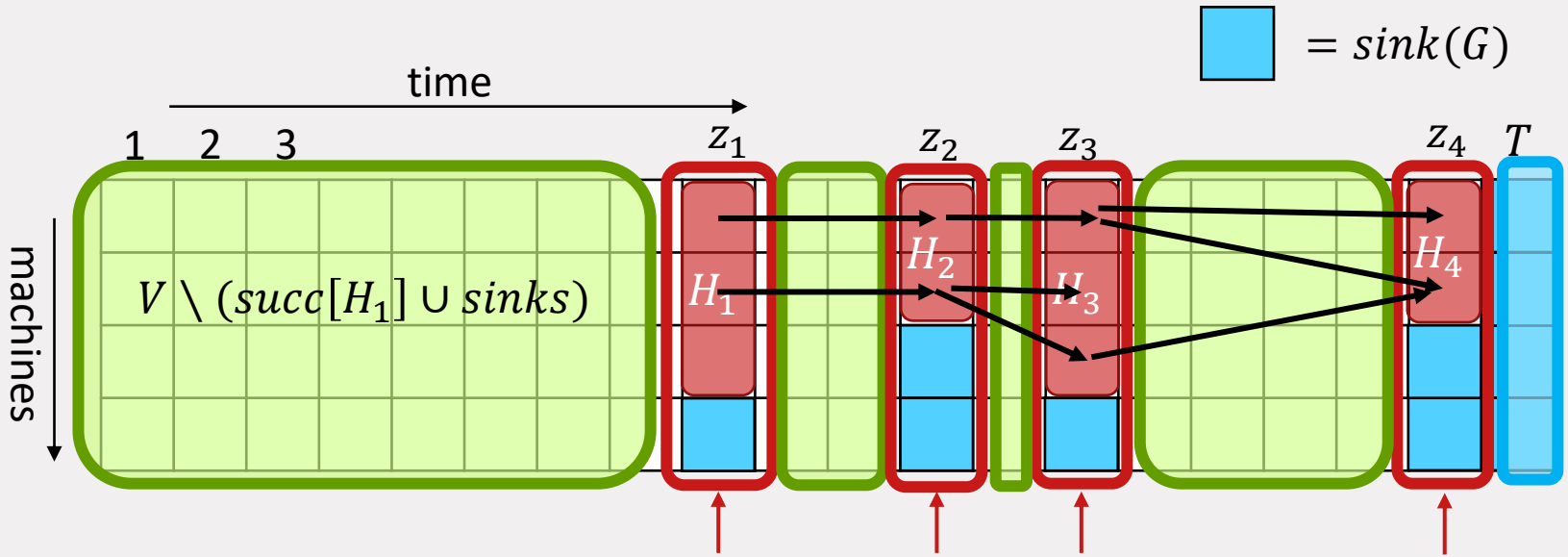

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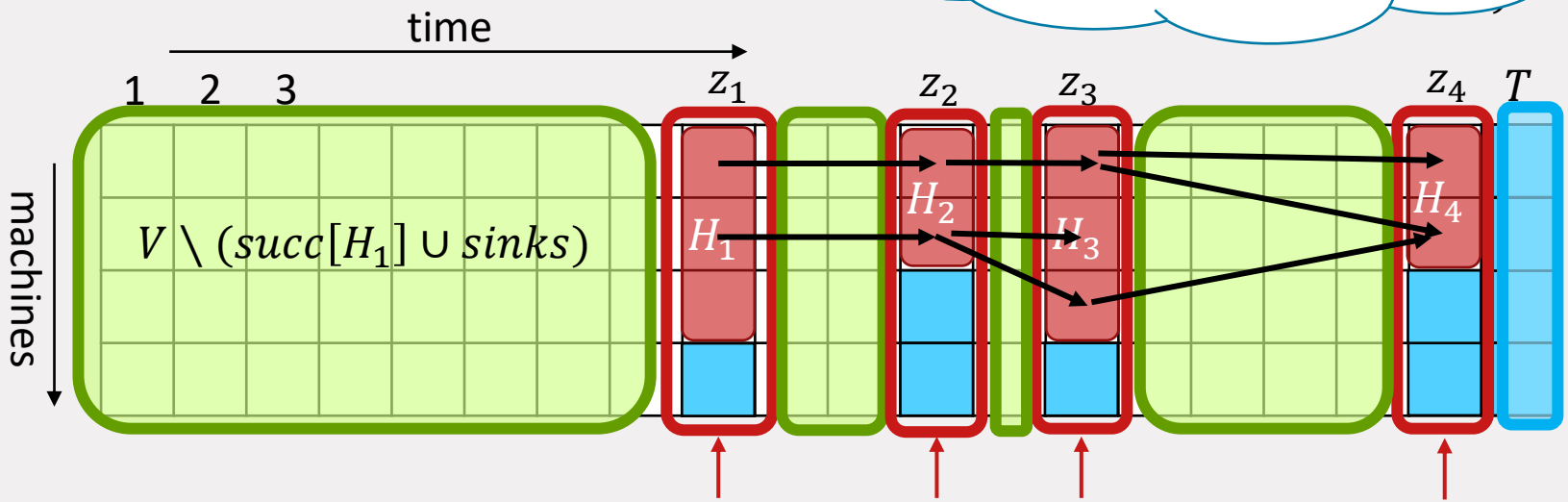
Sink-Adjusted Schedule

How to use this…?



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