TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

A Subexponential Time Algorithm for Makespan Scheduling of Unit Jobs with Precedence Constraints

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List of Open Problems by Garey and Johnson 1979

1. Graph Isomorphism
2. Subgraph Homeomorphism
3. Graph genus
4. Chordal graph completion
5. Chromatic index
6. Spanning tree parity problem
7. Partial order dimension
8. Precedence constrained 3-processor scheduling
9. Linear Programming
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11. Composite number
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$2^{O\left((\log n)^3\right)}$ time
[Babai 2017]

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$2^{O\left(\sqrt{n}\cdot\log n\right)}$ time
This talk

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Literature overview $Pm|prec, p_j = 1|C_{\max}$

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Before:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $O\left(2^n \cdot \binom{n}{m}\right)$ time.

Our Result

Our result:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $\left(1 + \frac{n}{m}\right)^{O(\sqrt{nm})}$ time.

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Key points in algorithm:

1. New decomposition of schedules
2. Use of look-up table
3. Use of Dynamic Programming in combining results

$Pm|prec, p_j = 1|C_{\max}$

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m identical parallel machines

$m = 2$

$Pm|prec, p_j = 1|C_{\max}$

Given:

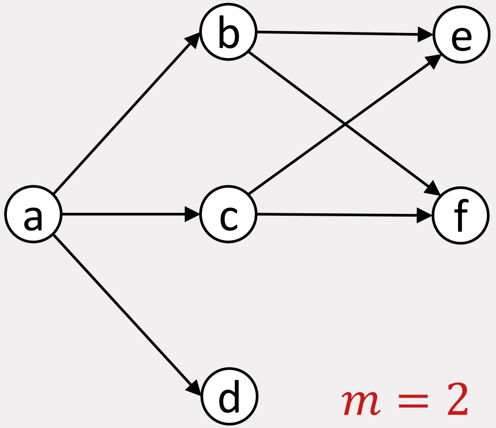
- n jobs of length 1

$m = 2$

$Pm|prec, p_j = 1|C_{\max}$

Given:

- n jobs of length 1
- A precedence graph G



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Given:

- n jobs of length 1
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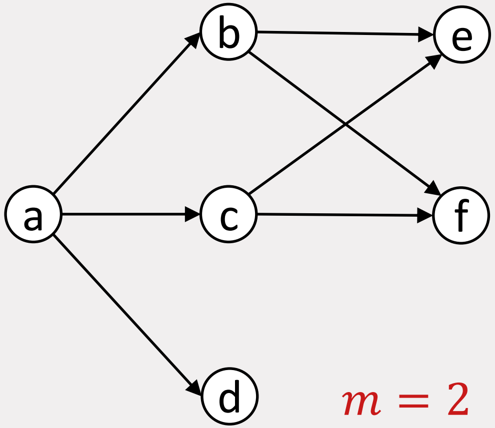
- $T \in \mathbb{N}$

Q: Is there a schedule of makespan T?



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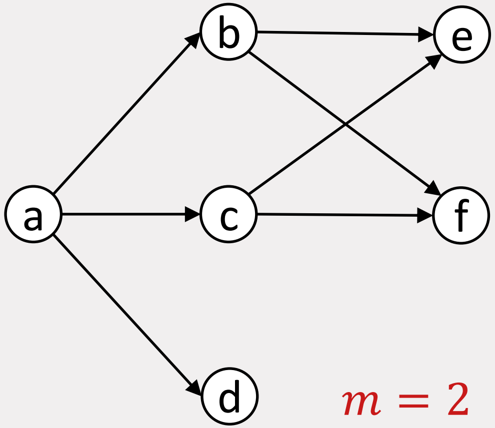

$m = 2$

time →

	1	2	3	4
1	a	b	d	f
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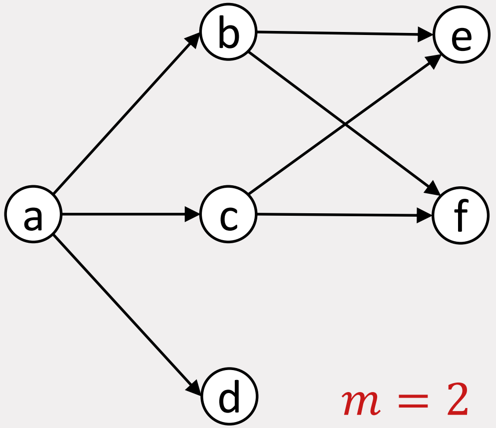

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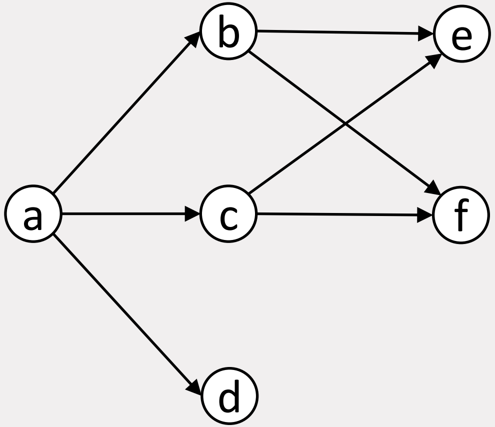
Observation:
Jobs of length one ⇒ `timeslots’

Definitions

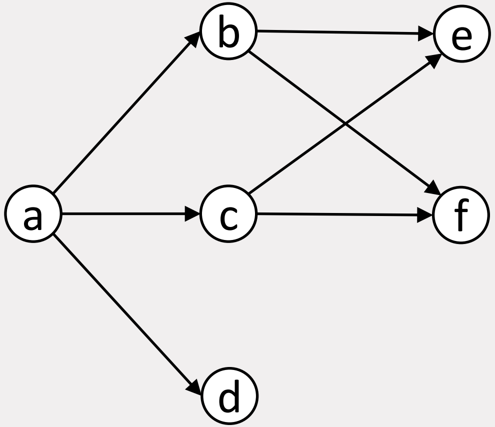


Precendence Constraints Graph G

Definitions

$G \Rightarrow$ partial order:

- $i \prec j$ if $(i, j) \in G$



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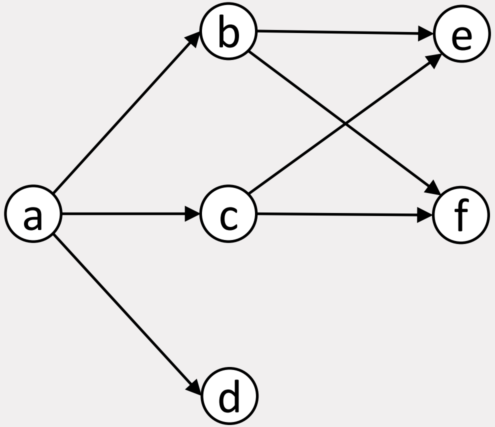

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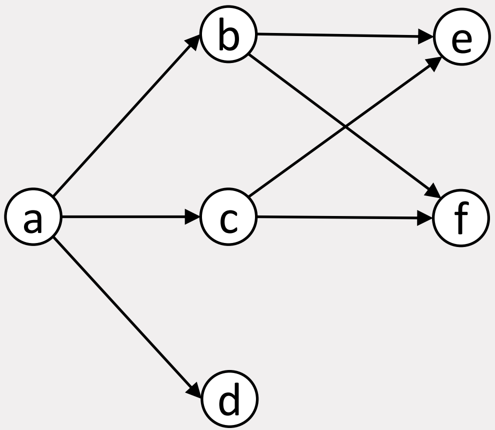

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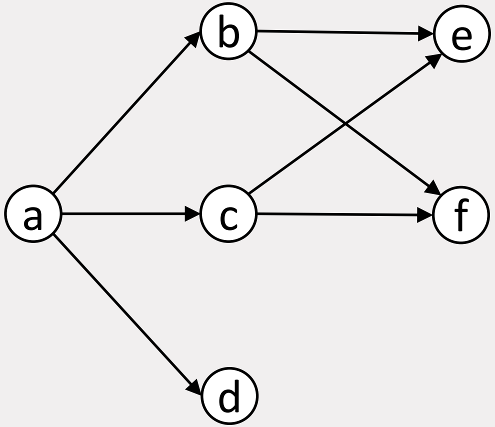

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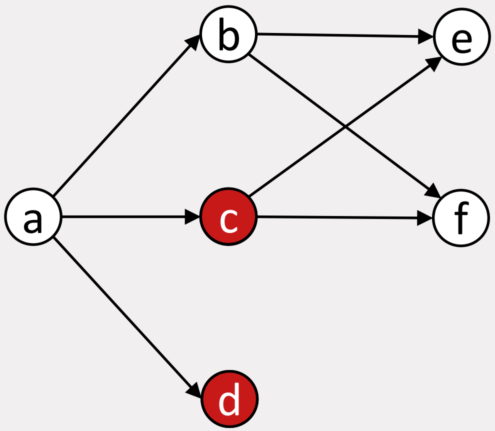

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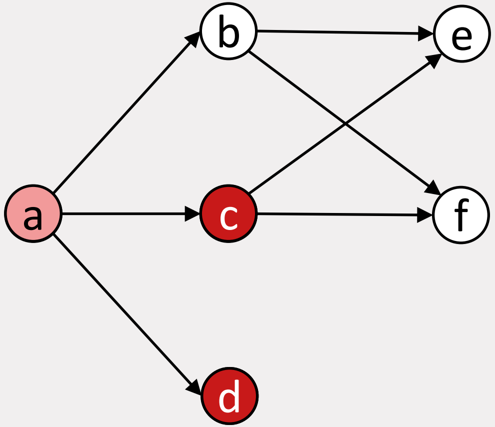

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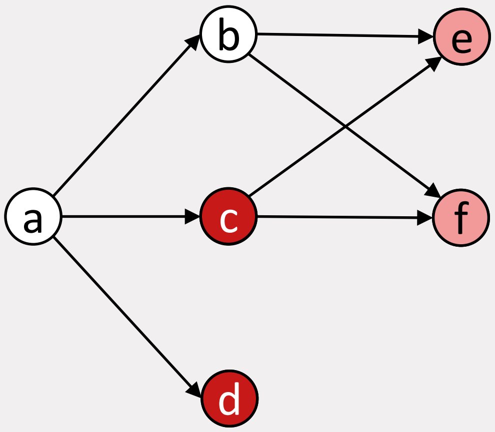

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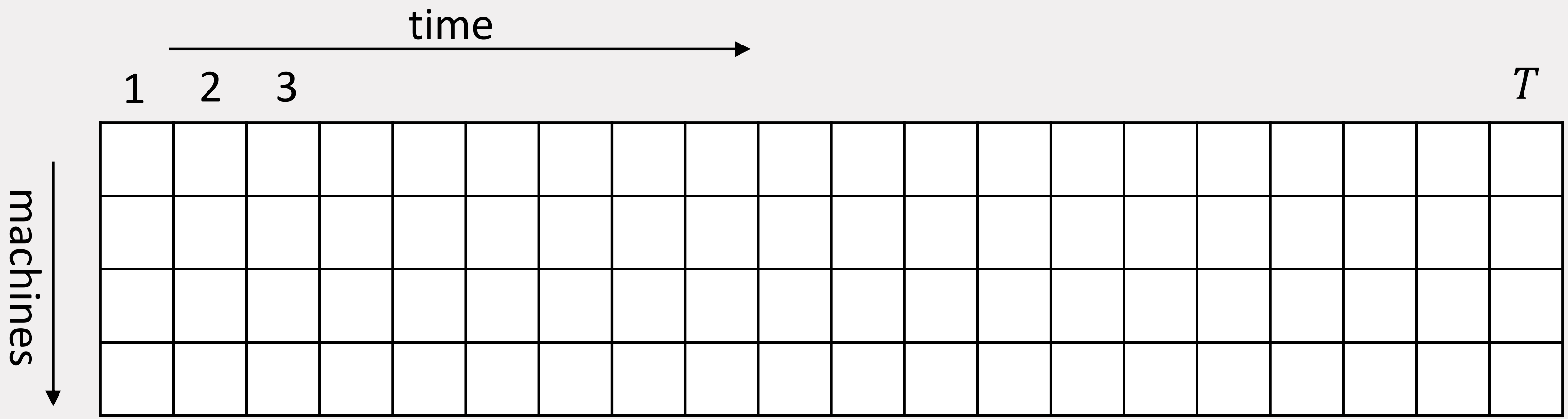
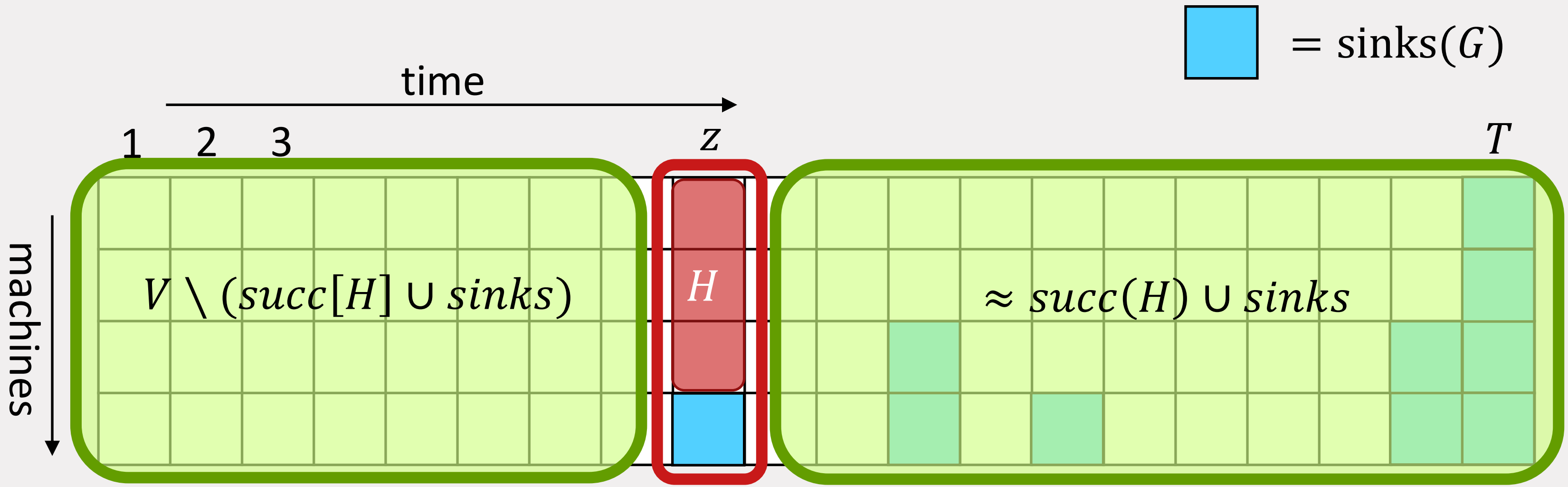
Zero-Adjusted Schedule (D&W)



Zero-Adjusted Schedule (D&W)

Assumption: $n = m \cdot T$

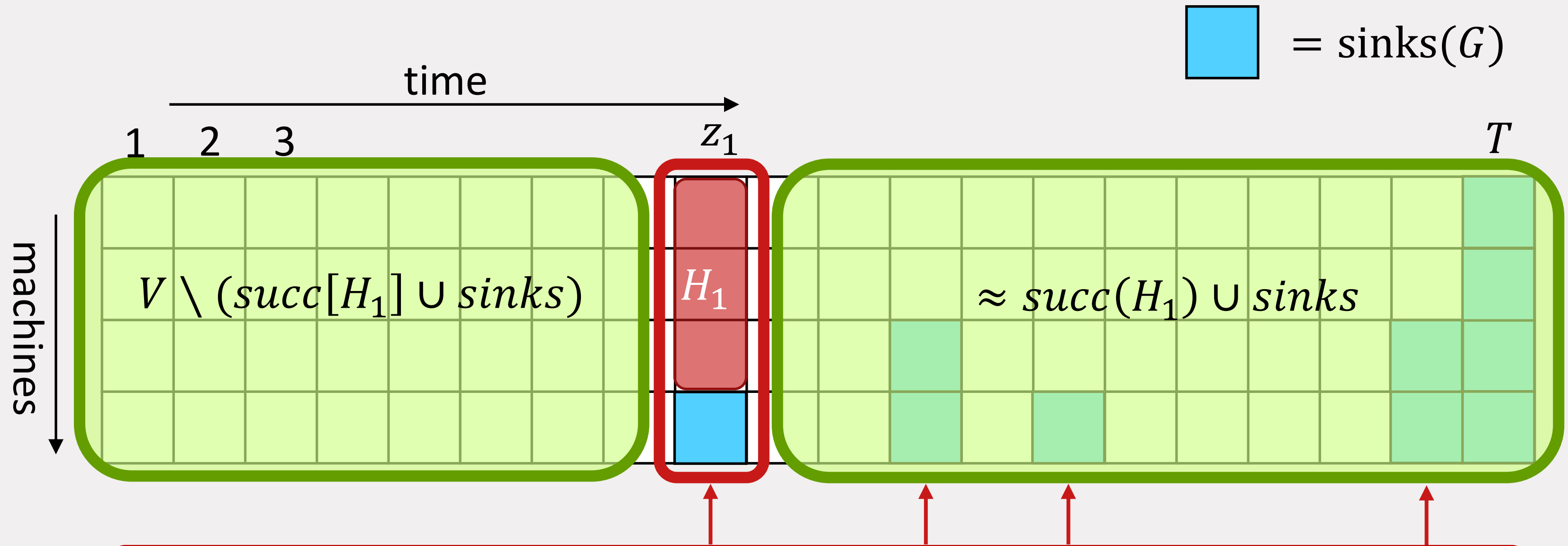


Let $z \in [1, T]$ be the first moment with a sink.

<u>D&W:</u> W.m.a. Each job x **after z** is a sink or a successor of a job at time z.

Sink-Adjusted Schedule

Assumption: $n = m \cdot T$



$z \in [1, T]$ is a ***sink moment*** if there are both sinks and non-sinks at time z.

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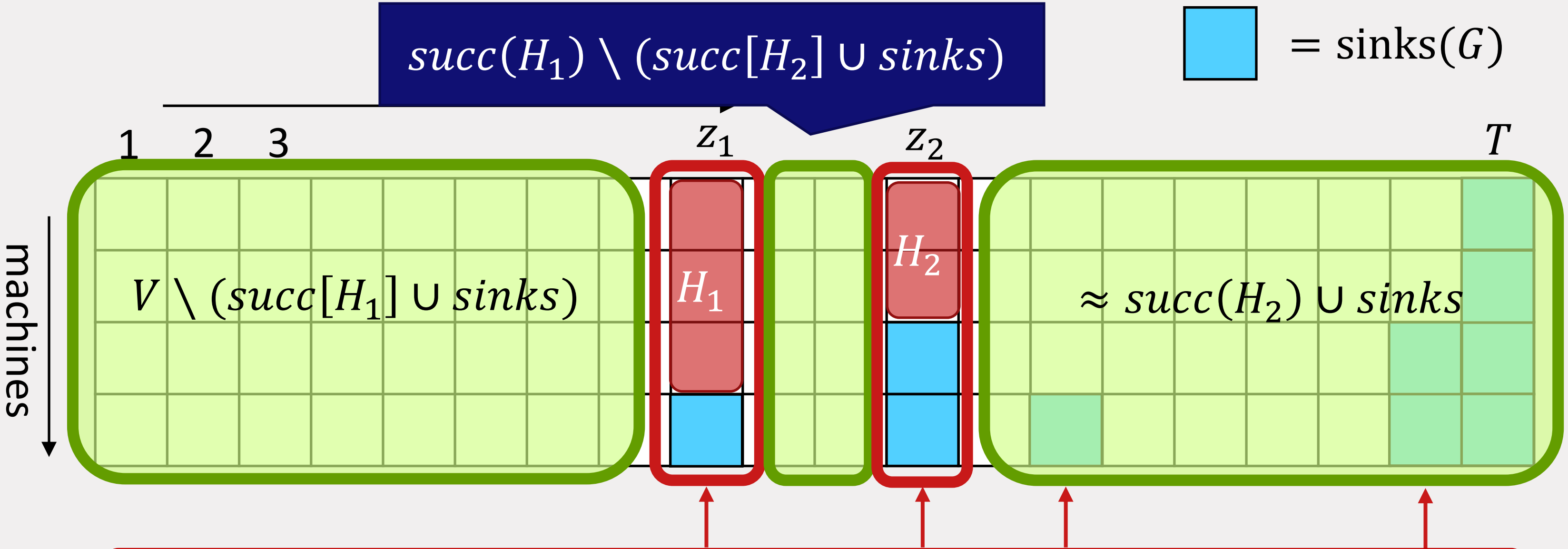

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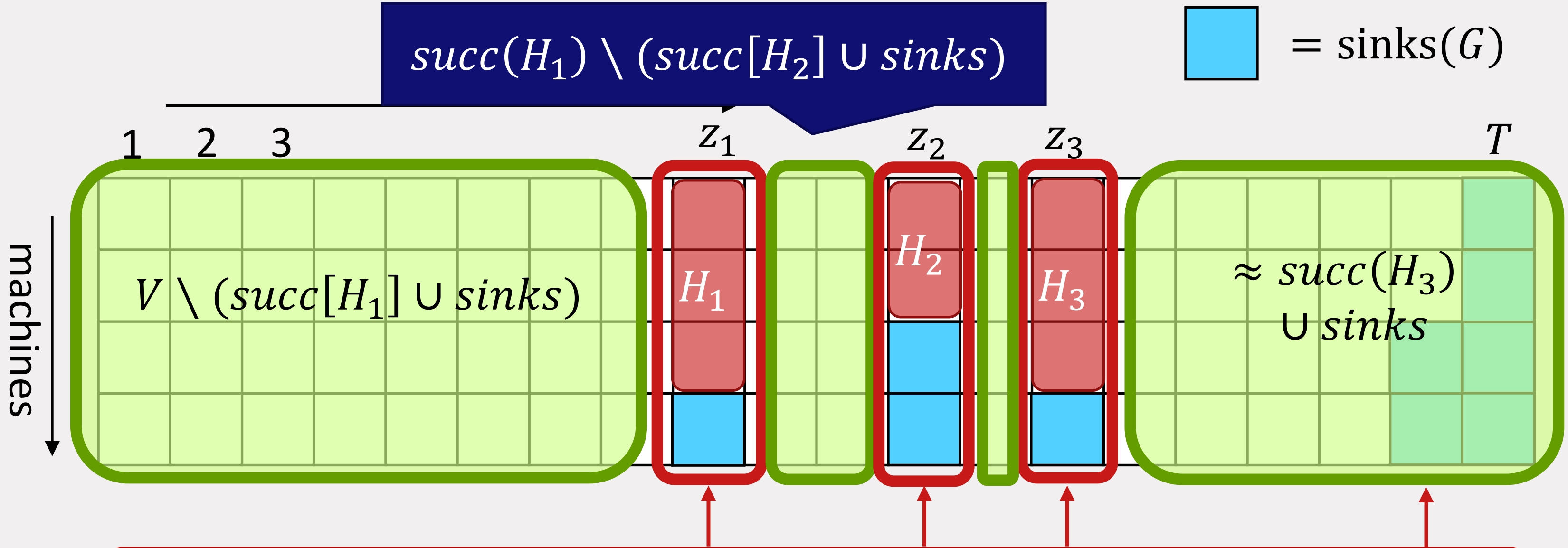

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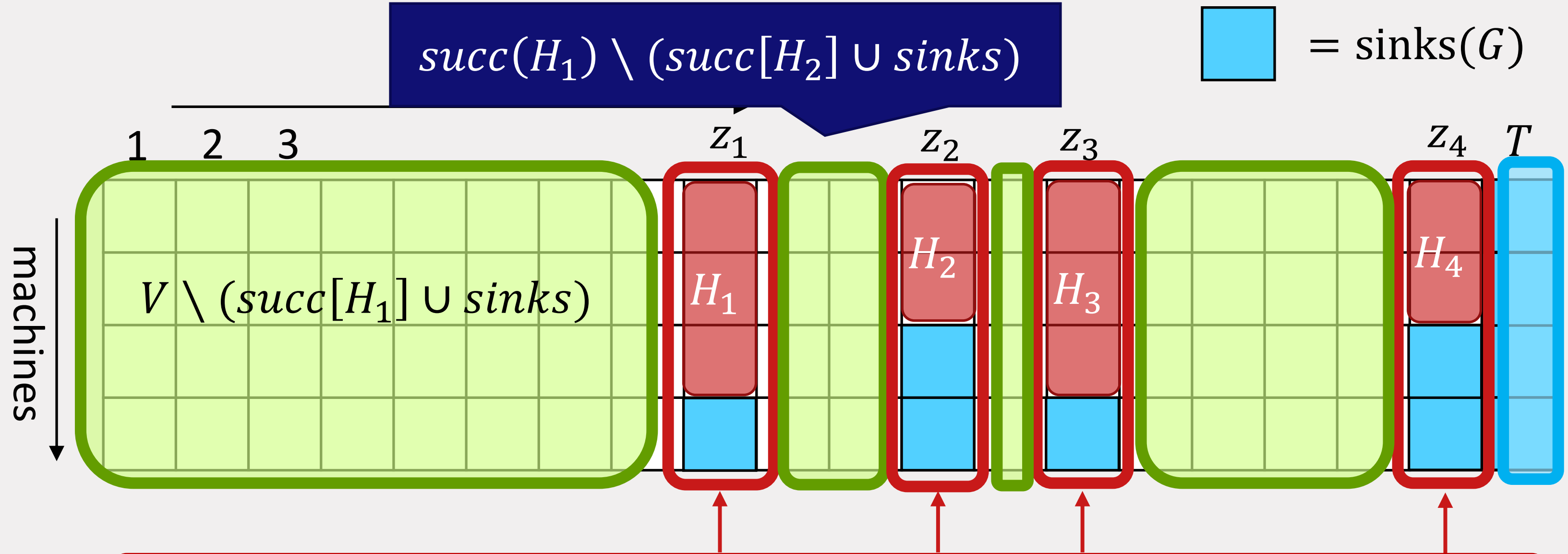

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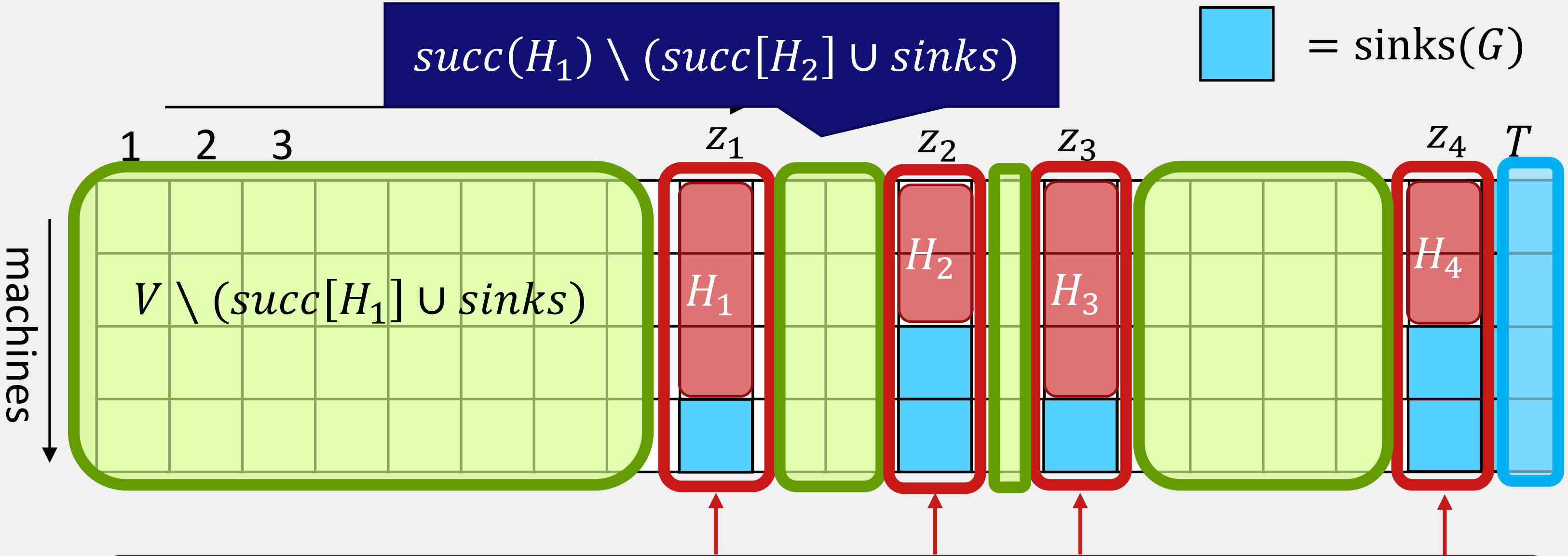

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No sinks in

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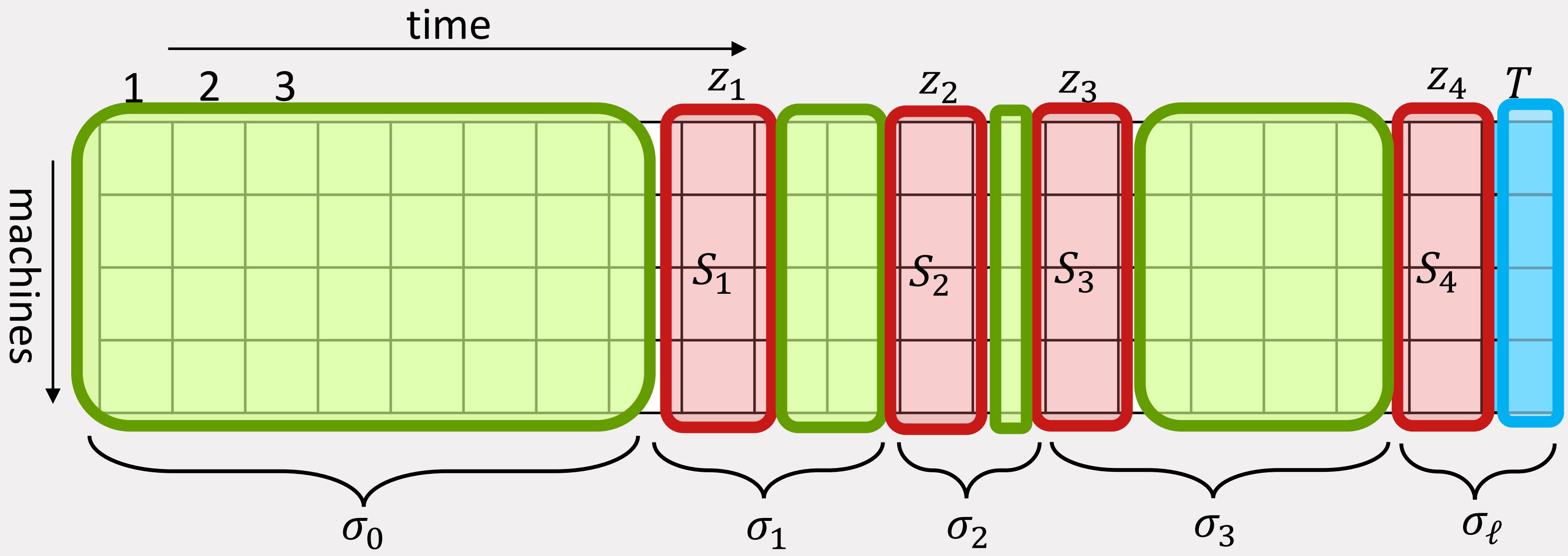

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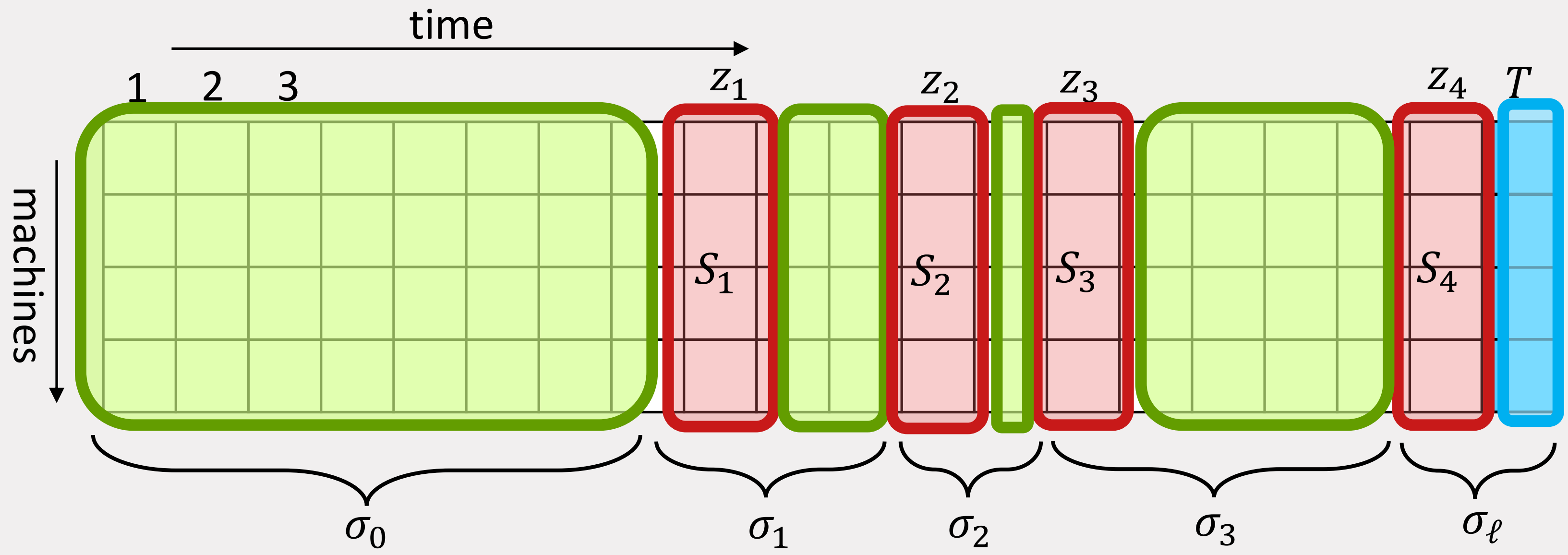


$$\sigma_i = \sigma(S_i, S_{i+1})$$

Summary of Algorithm

Input: X a set of jobs, m

Output: minimal makespan of X

1. For each $S_1, S_2 \in \binom{X}{\leq m}$ do:
   a) Compute recursively makespan of $X \cap \sigma(S_1, S_2)$
2. Find a minimal combination of these subschedules
3. Return found schedule

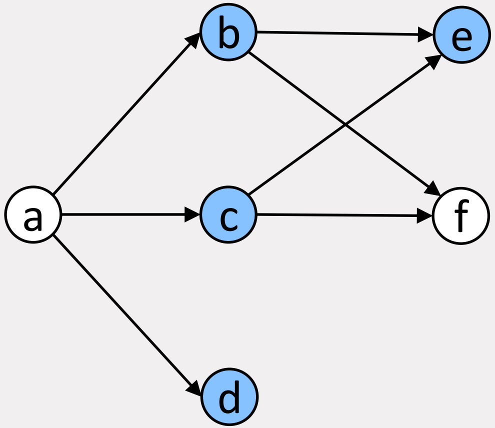
Simple analysis: $2^n\ poly(n)$ time…?

Only n^{2m} subproblems

Use of DP in $\approx n^{2m}$ time

Describing subschedules

Let σ be a *subschedule*.



time

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Describing subschedules

Let σ be a *subschedule*.

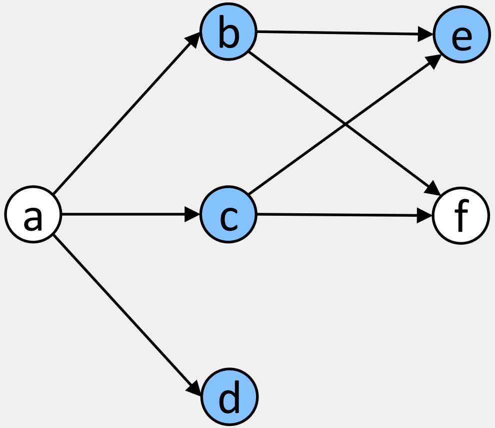
Jobs $V(\sigma)$ can be described as

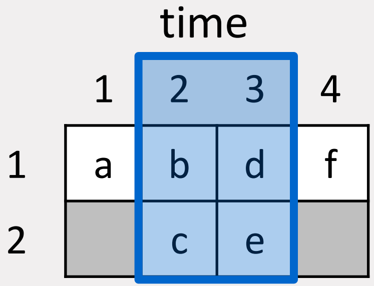
$$V(\sigma) = \mathrm{succ}[A] \cap \mathrm{pred}[B]$$

where

A = minimal elements = sources

B = maximal elements = sinks





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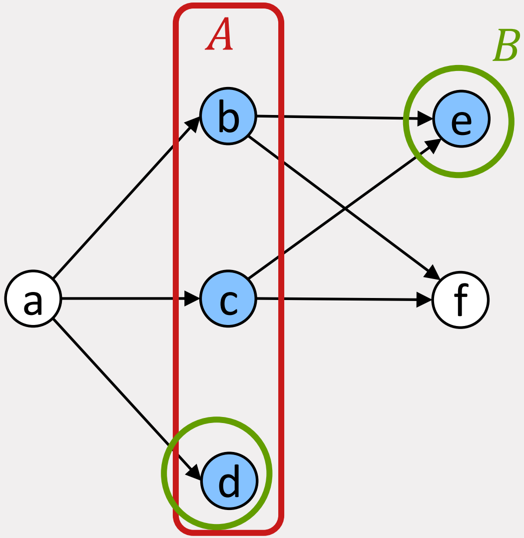

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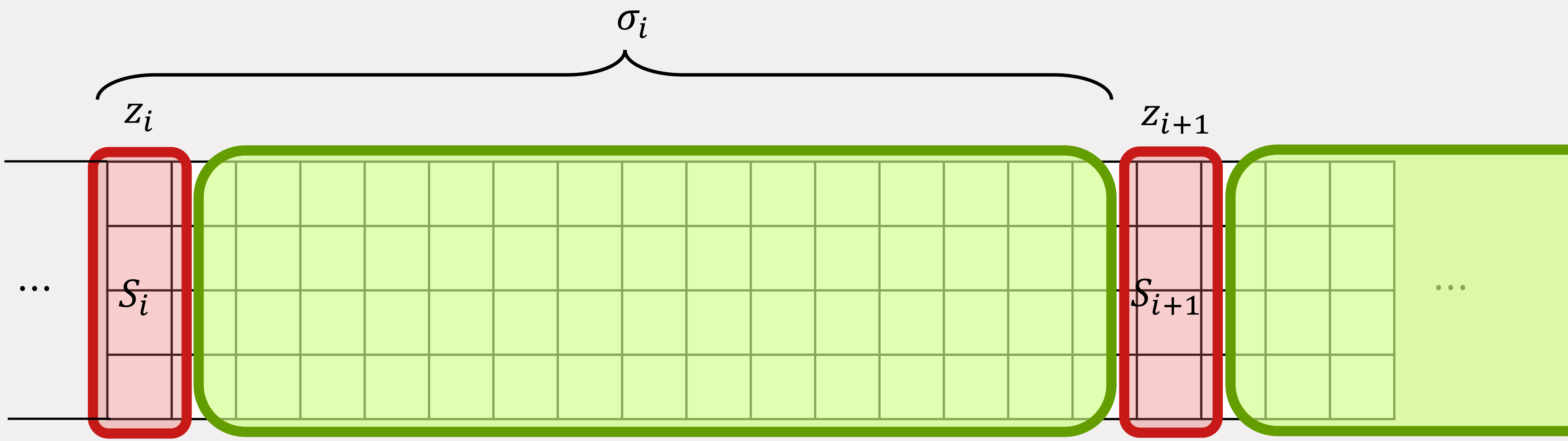

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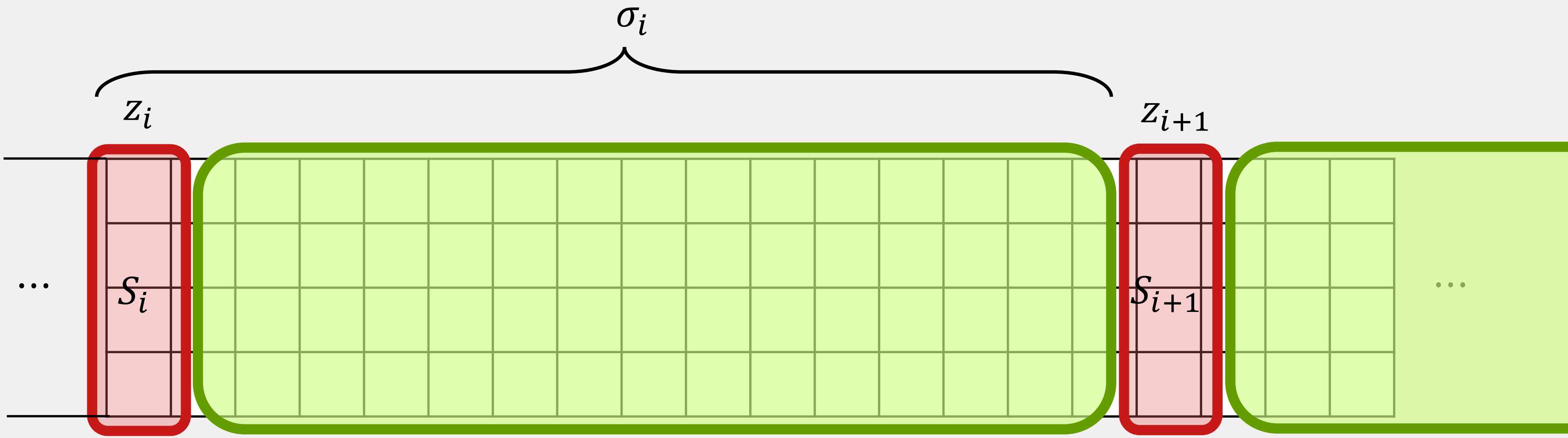

Lemma:

$$V(\sigma_i) \approx \mathrm{succ}[S_i] \setminus \left(\mathrm{succ}[S_{i+1}] \cup \mathrm{sinks}(G)\right)$$

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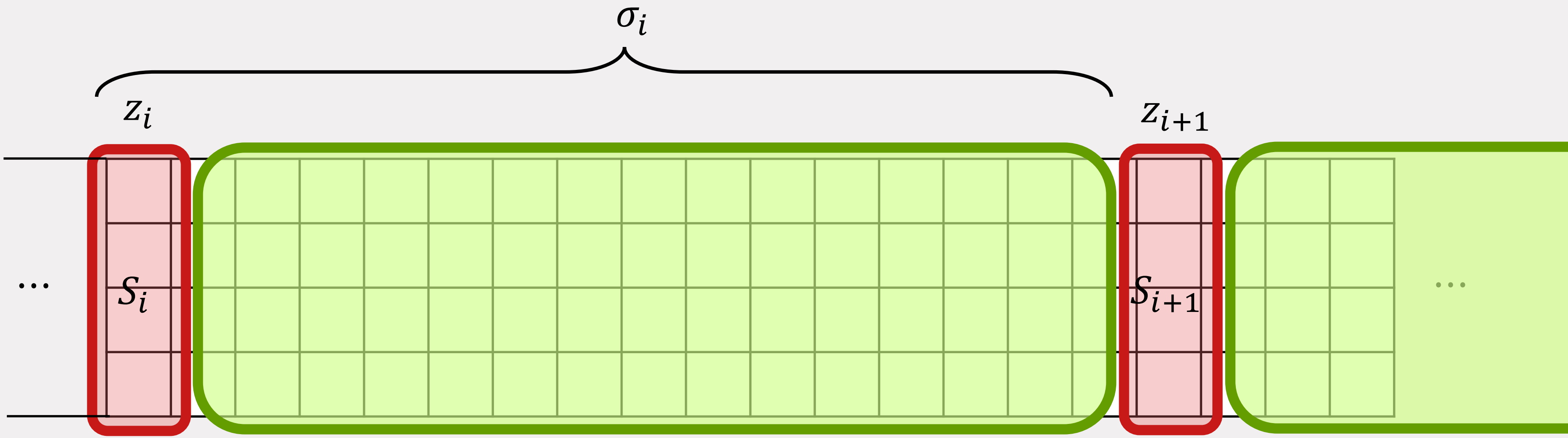

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Algorithm

Store subprohlem results in lookup table:

$$T(A, B) \coloneqq \text{min makespan of succ}[A] \cap \text{pred}[B]$$

Use of lookup table, how many *different* subproblems?

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Remark: $|S_i| \leq m$!!

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Case $\lvert B\rvert \leq \sqrt{n}$	Case $\lvert B\rvert > \sqrt{n}$
$\Rightarrow$ only $\binom{n}{\sqrt{n}} = 2^{O(\sqrt{n} \cdot \log n)}$ different B's	

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$\Rightarrow$ only $\binom{n}{\sqrt{n}} = 2^{O(\sqrt{n} \cdot \log n)}$ different B's ✓	In next step: make $\sqrt{n}$ jobs progress!

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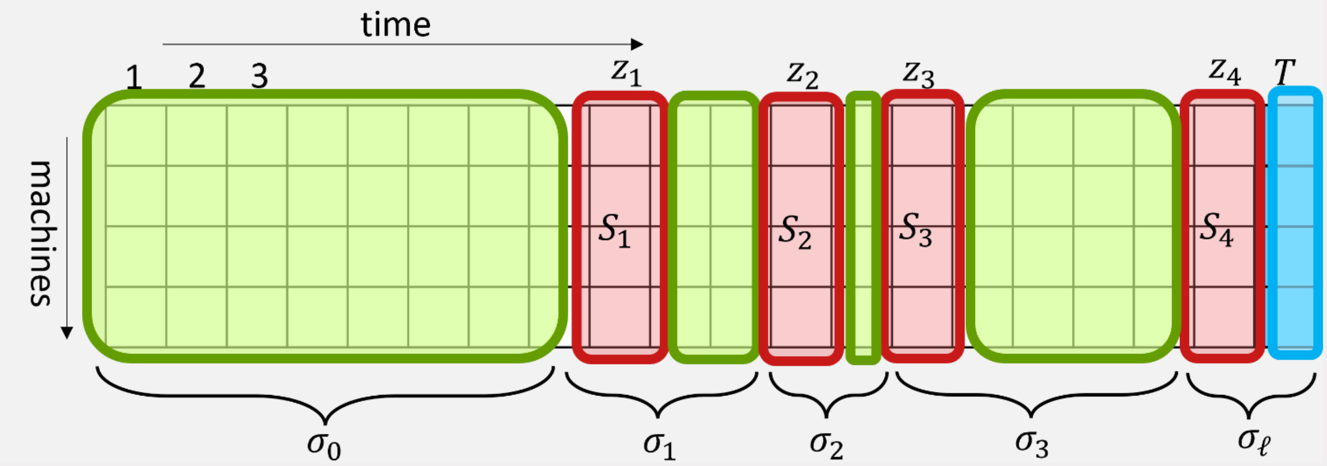
Algorithm

No sinks in [green box]

$V(\sigma) = \text{succ}[A] \cap \text{pred}[B]$



Case $\lvert B\rvert \leq \sqrt{n}$	**Case** $\lvert B\rvert > \sqrt{n}$
$\Rightarrow$ only $\binom{n}{\sqrt{n}} = 2^{O(\sqrt{n}\cdot\log n)}$ different B's ✓	In next step: make $\sqrt{n}$ jobs progress!

Algorithm

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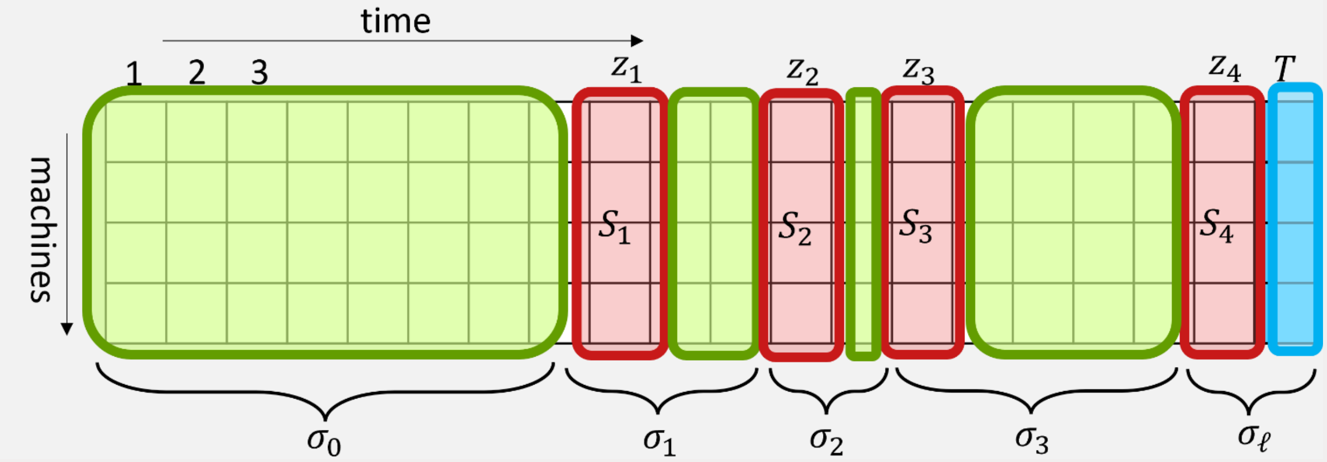

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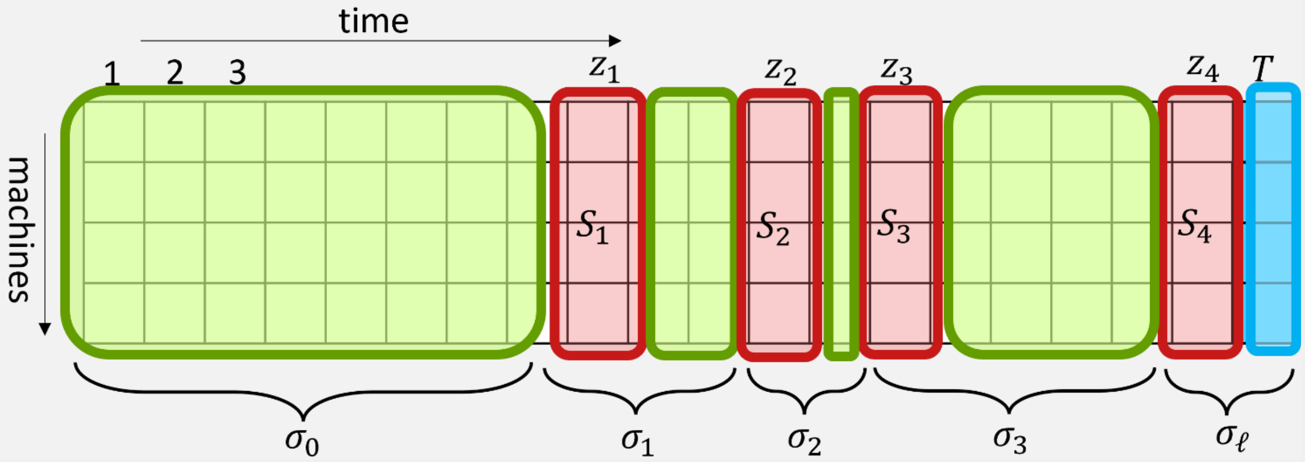

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$|\sigma_i| \leq n - \sqrt{n} + m$

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TU/e

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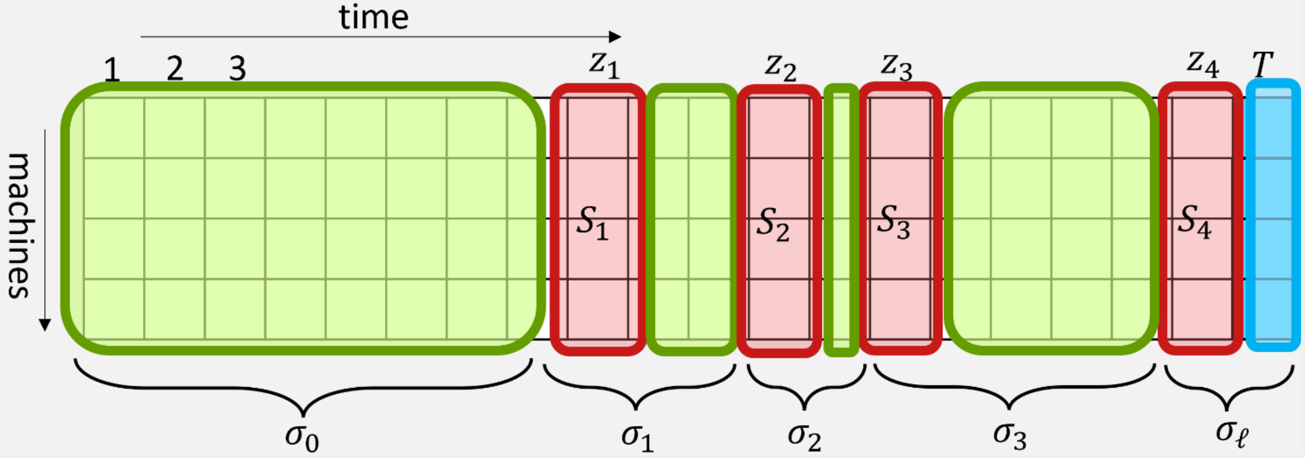

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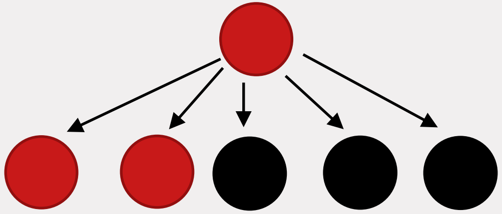
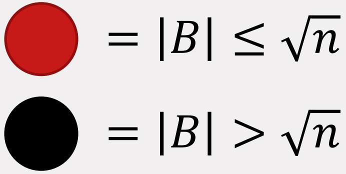
$\leq n^{2m} = n^{O(1)}$
children





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$\leq n^{2m} = n^{O(1)}$
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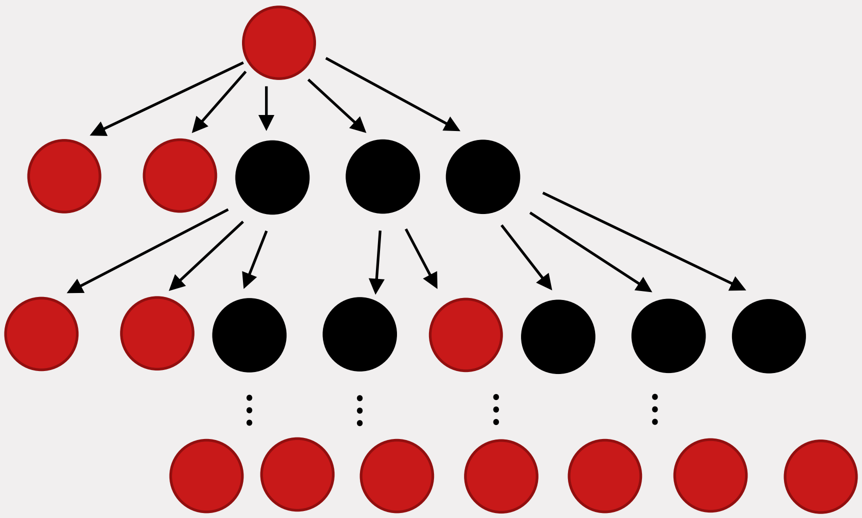
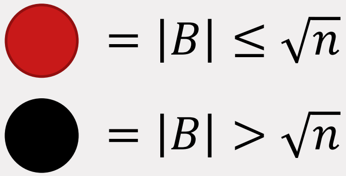
Height $\leq \sqrt{n}$

$\leq \left(n^{O(1)}\right)^{\sqrt{n}}$
$= 2^{O(\sqrt{n} \cdot \log n)}$
nodes in tree!





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Algorithm

Store subproblem results in lookup table:

$$T(A, B) \coloneqq \text{min makespan of succ}[A] \cap \text{pred}[B]$$

Use of lookup table, how many *different* subproblems?

$\Rightarrow$ at most $2^{O(\sqrt{n} \cdot \log n)}$ trees with $2^{O(\sqrt{n} \cdot \log n)}$ nodes

$\Rightarrow$ $2^{O(\sqrt{n} \cdot \log n)}$ *different* subproblem!

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Summary of Algorithm

Input: *A, B, m*

Output: minimal makespan of $\mathrm{succ}[A] \cap \mathrm{pred}[B] = \mathrm{Jobs}$

1. For each $S_1, S_2 \in \binom{\mathrm{Jobs}}{\leq m}$ do:
   a) Compute recursively makespan of $\mathrm{Jobs}\ \cap (\mathrm{succ}[S_1] \setminus (\mathrm{succ}[S_2] \cup B)$
2. Find a minimal combination of these subschedules
3. Return found schedule

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Conclusion

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Main result:

$Pm|prec, p_j = 1|C_{\max}$ in $2^{O(\sqrt{n} \cdot \log n)}$ time.

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- New decomposition of schedules
- Use of look-up table
- Use of Dynamic Programming in combining results

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Future Research:

$P3|prec, p_j = 1|C_{\max}$ in quasi-polynomial time?

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Thanks for your attention!