A Subexponential Time Algorithm for Makespan Scheduling of Unit Jobs with Precedence Constraints

Jesper Nederlof

UU

Céline Swennenhuis

TU/e

Karol Węgrzycki

Saarland University

Department of Mathematics and Computer Science, Eindhoven University of Technology

$$Pm|prec, p_j = 1|C_{\max}$$

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m identical parallel machines

$m = 2$

$Pm|prec, p_j = 1|C_{\max}$

Given:

- n jobs of length 1

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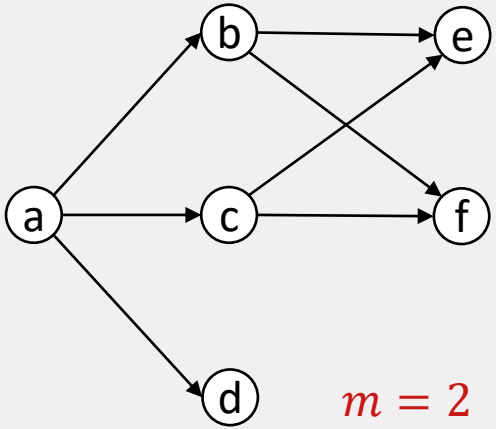
- A precedence graph G



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- $T \in \mathbb{N}$

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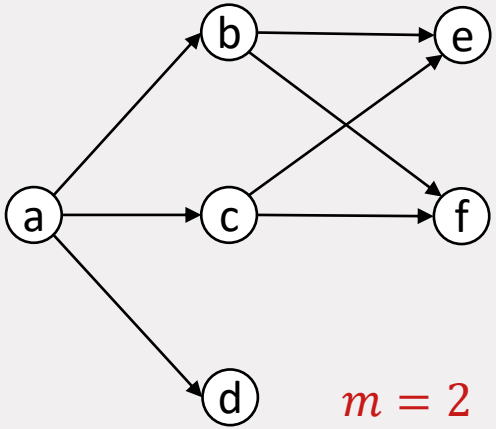

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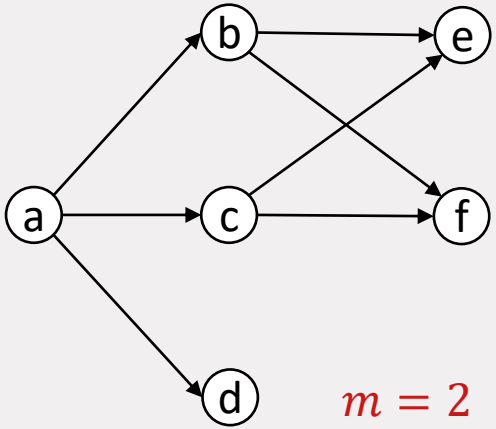

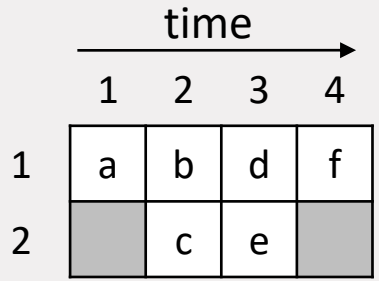

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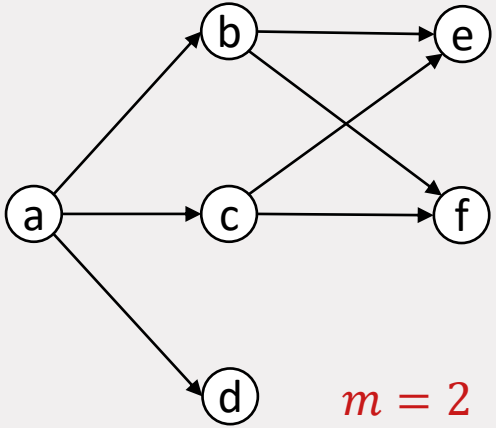

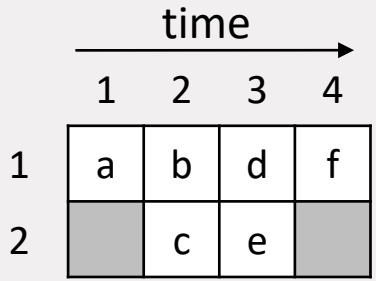

Observation:
Jobs of length one ⇒ `timeslots'

List of Open Problems by Garey and Johnson 1979

1. Graph Isomorphism
2. Subgraph Homeomorphism
3. Graph genus
4. Chordal graph completion
5. Chromatic index
6. Spanning tree parity problem
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8. Precedence constrained 3-processor scheduling
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$2^{O\left((\log n)^3\right)}$ time
[Babai 2017]

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$2^{O(\sqrt{n}\cdot\log n)}$ time
This talk

Literature overview $Pm|prec, p_j = 1|C_{\max}$

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Before:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $O\left(2^n \cdot \binom{n}{m}\right)$ time.

Our Result

Our result:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $\left(1 + \frac{n}{m}\right)^{O(\sqrt{nm})}$ time.

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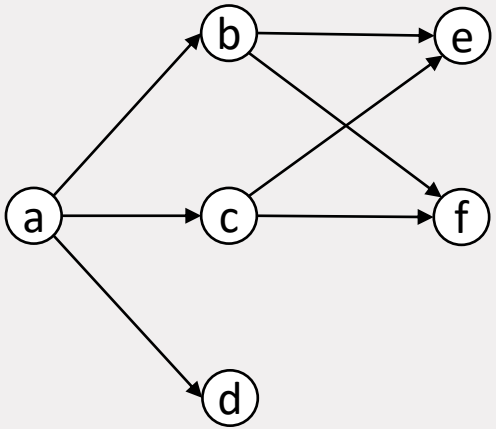
Key points in algorithm:

1. New decomposition of schedules
2. Use of look-up table
3. Use of Dynamic Programming in combining results

Definitions

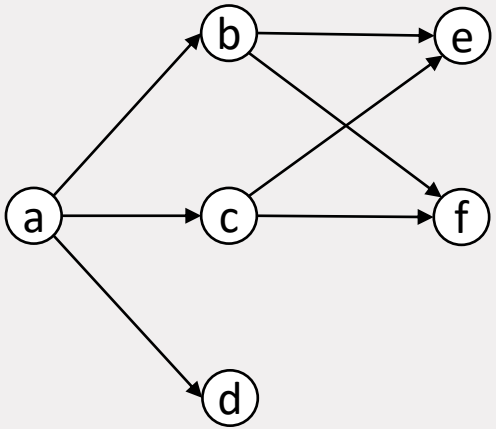


Precendence Constraints Graph G

Definitions

$G \Rightarrow$ partial order:

- $i \prec j$ if $(i, j) \in G$



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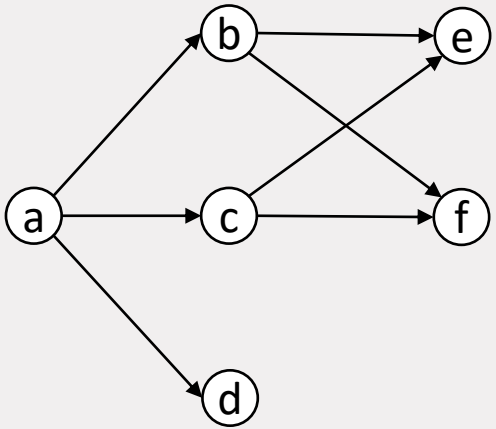

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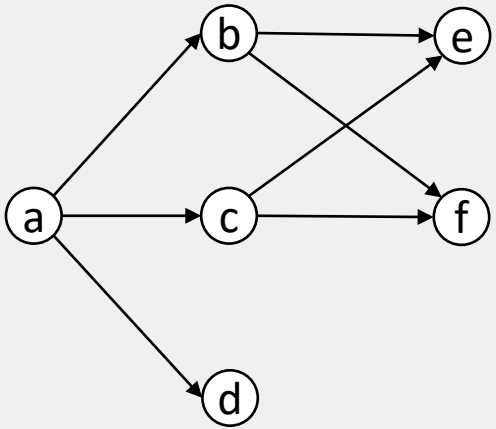

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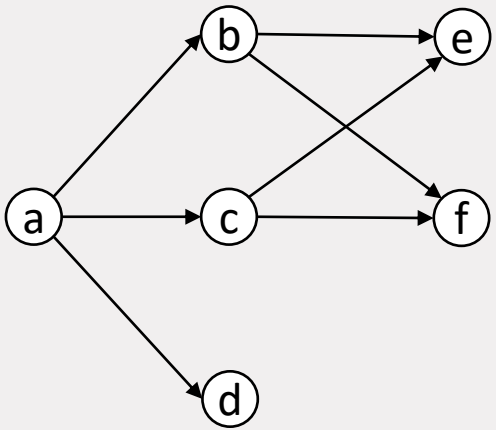

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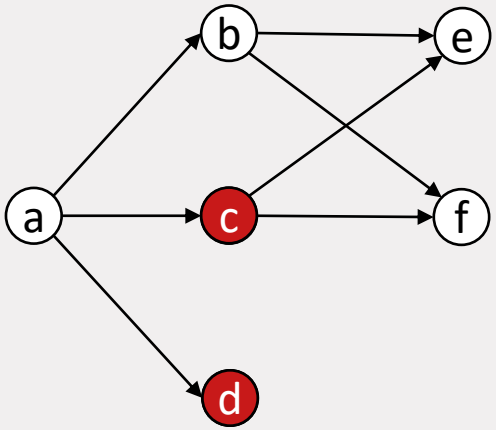

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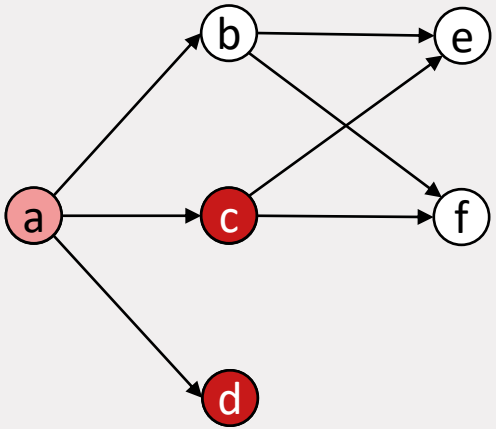

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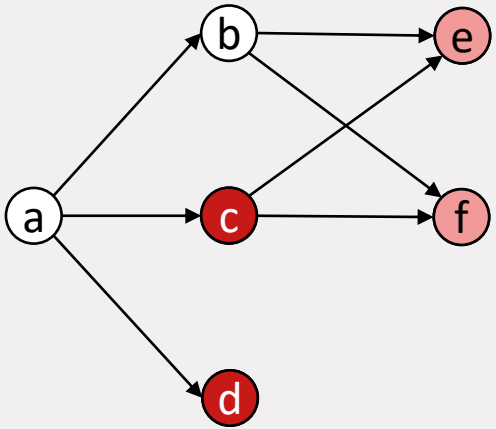

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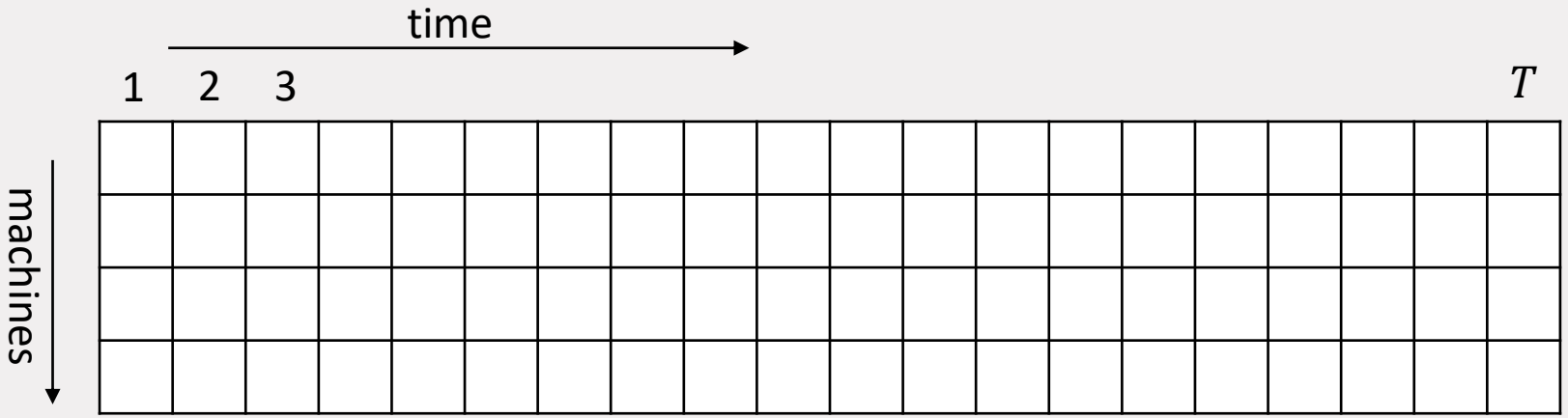
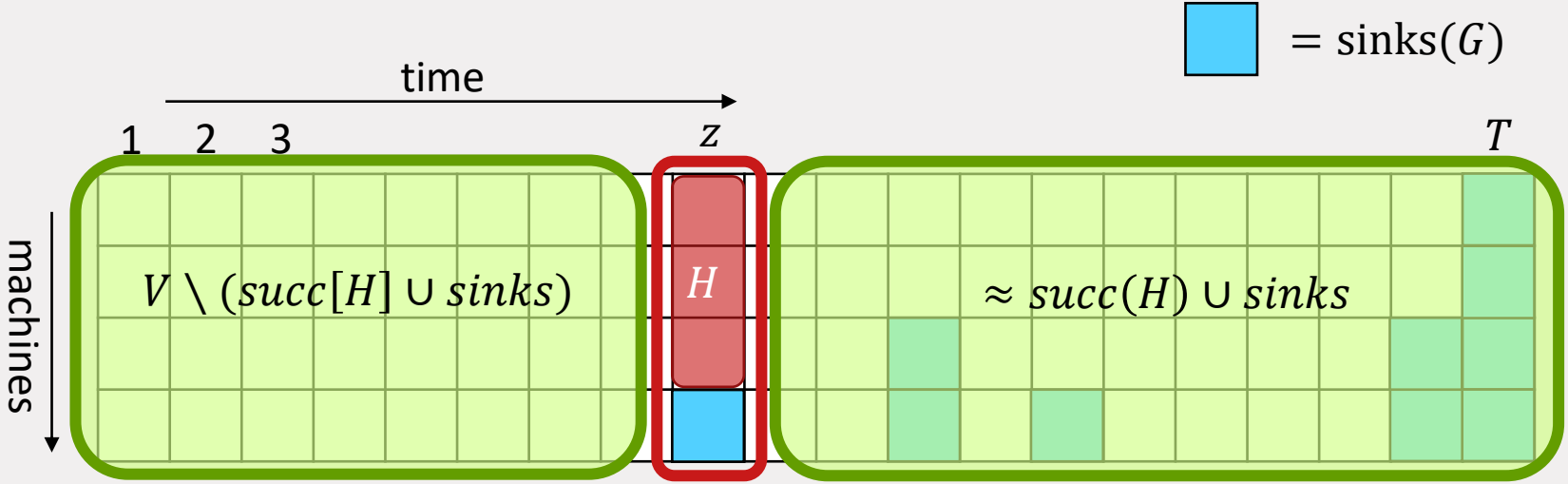
Zero-Adjusted Schedule (D&W)



Zero-Adjusted Schedule (D&W)

Assumption: $n = m \cdot T$



Let $z \in [1, T]$ be the first moment with a sink.

D&W: W.m.a. Each job x **after *z*** is a sink or a successor of a job at time z.

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Sink-Adjusted Schedule

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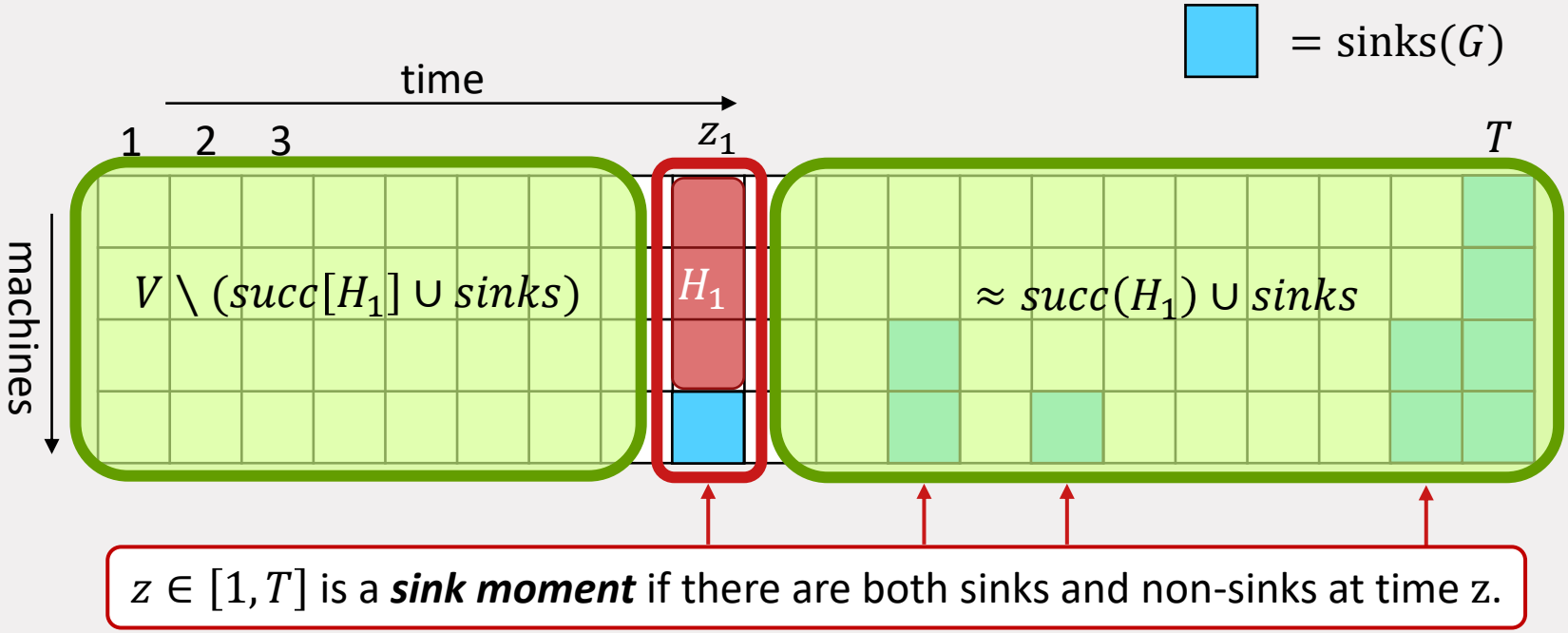

$z \in [1, T]$ is a ***sink moment*** if there are both sinks and non-sinks at time z.

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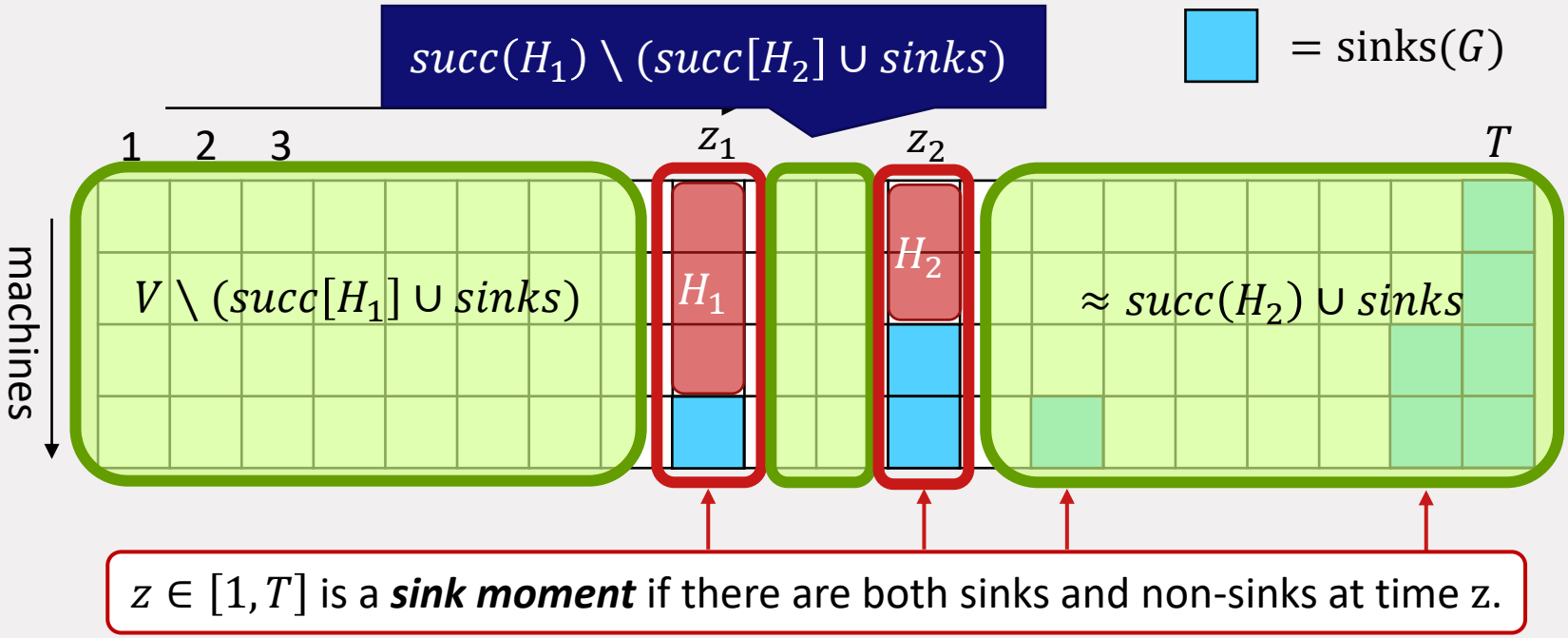

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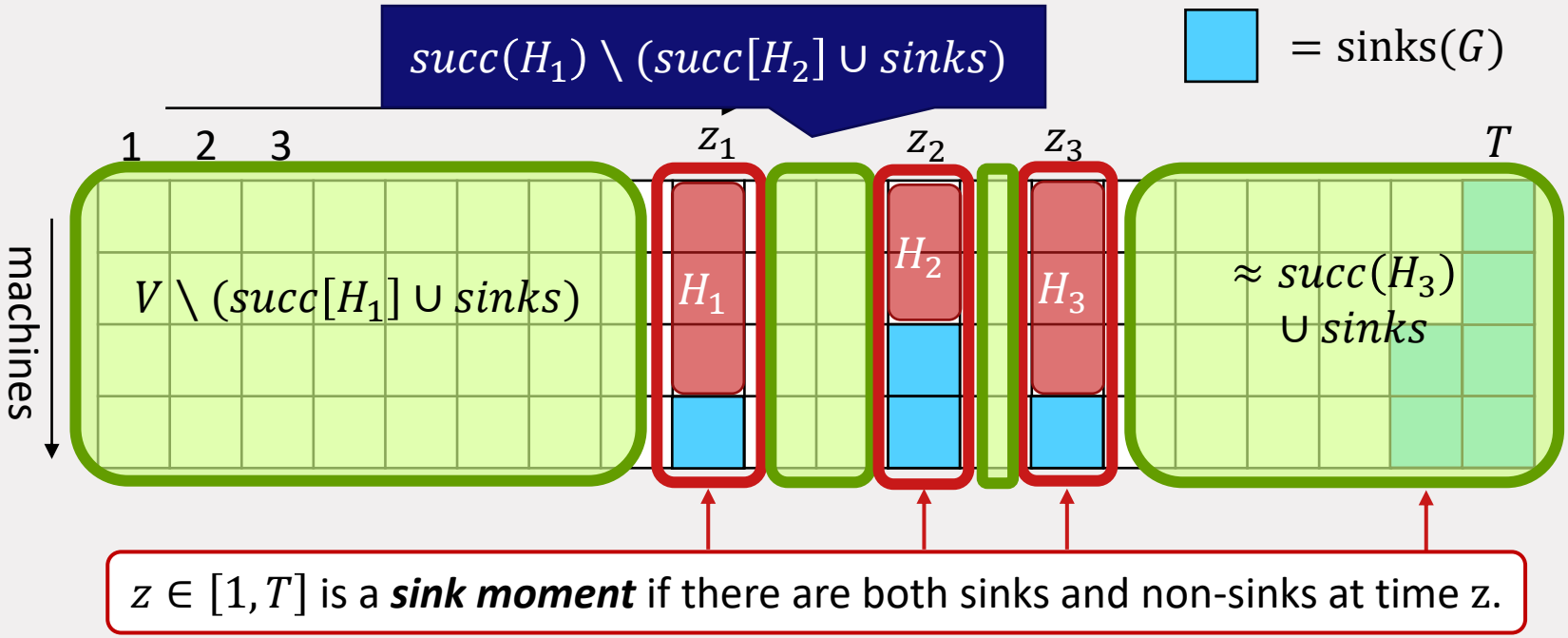

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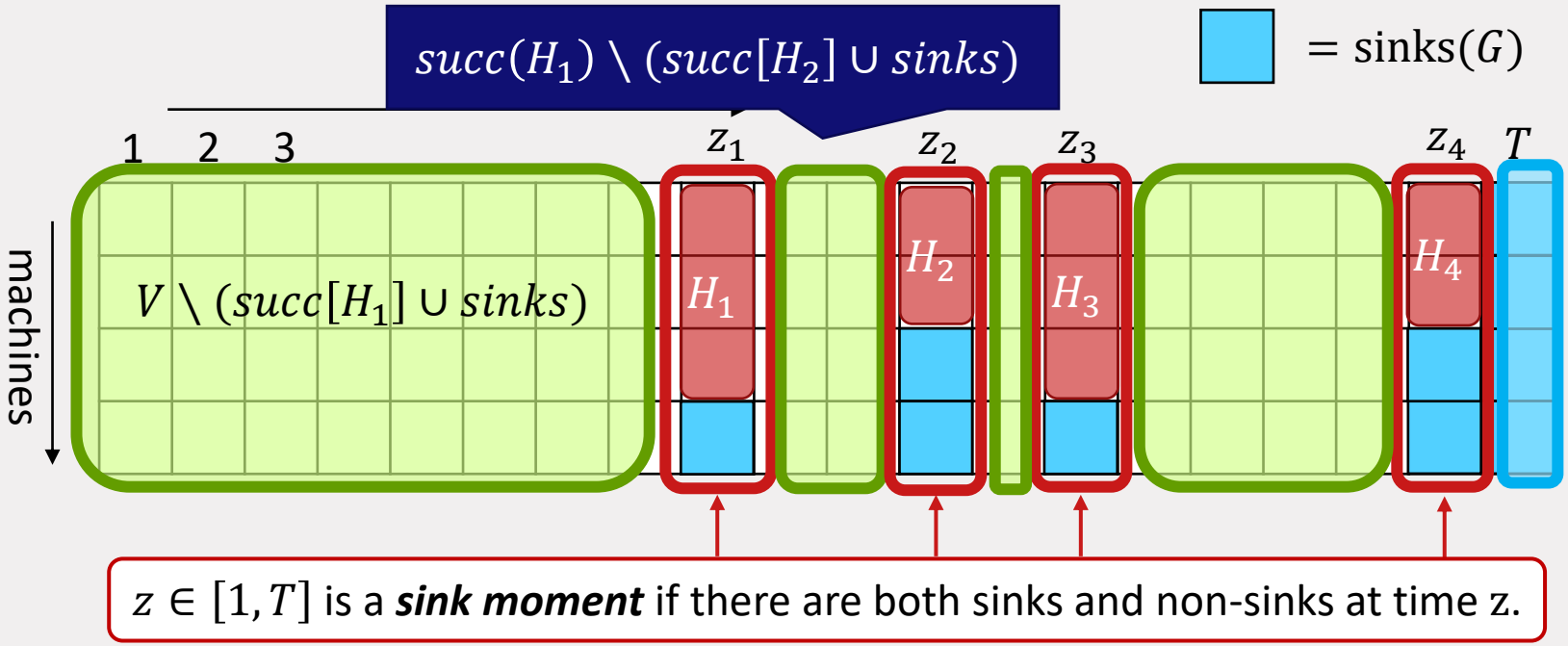

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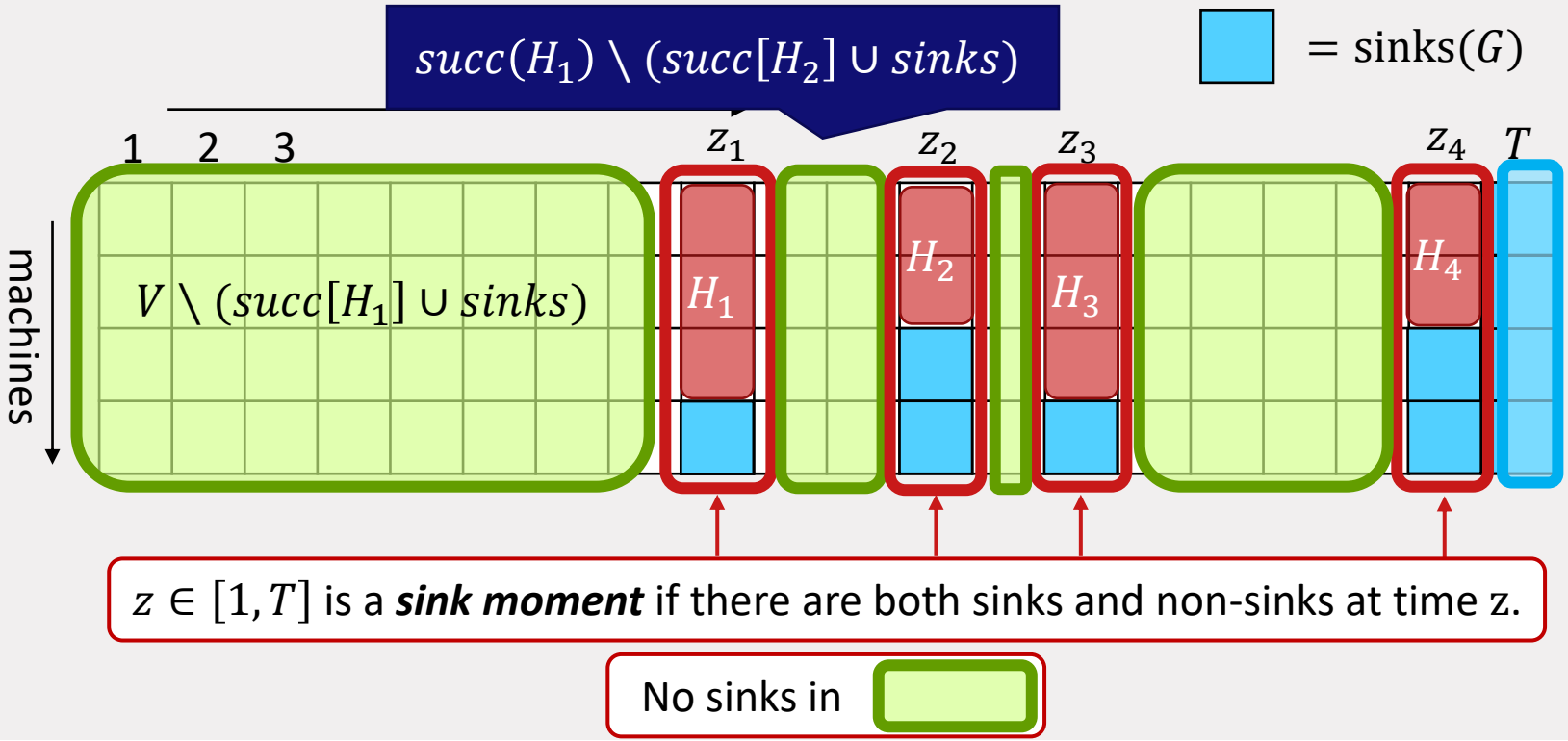

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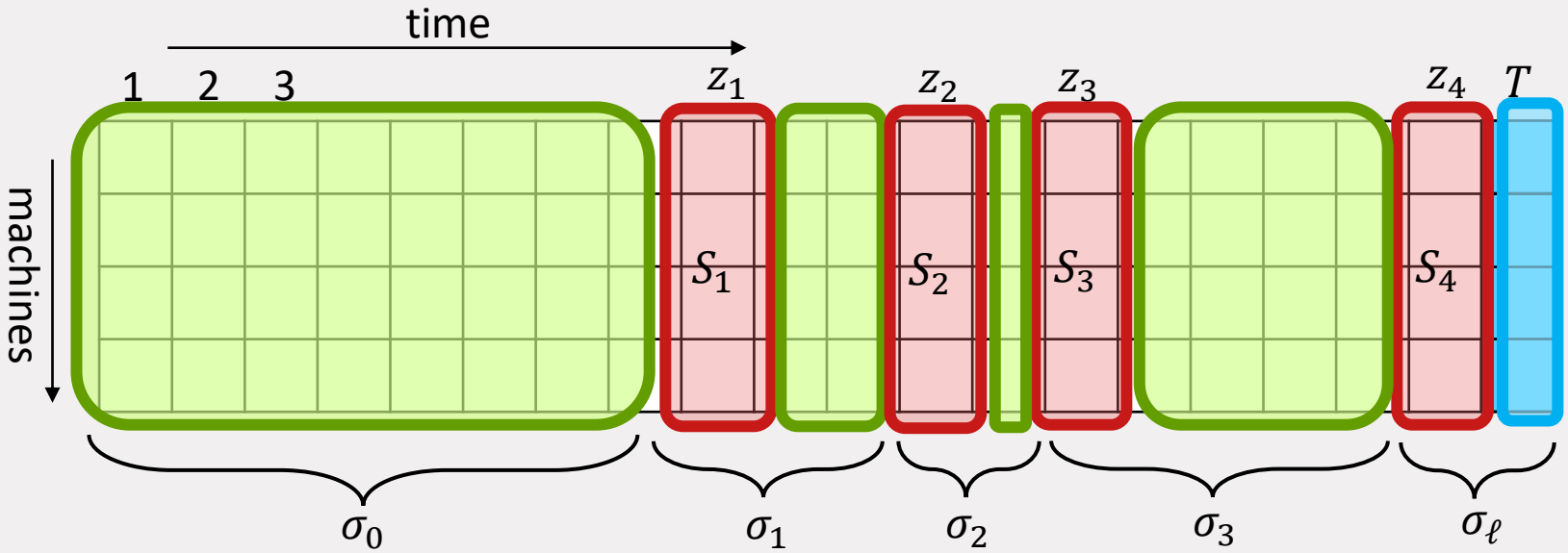

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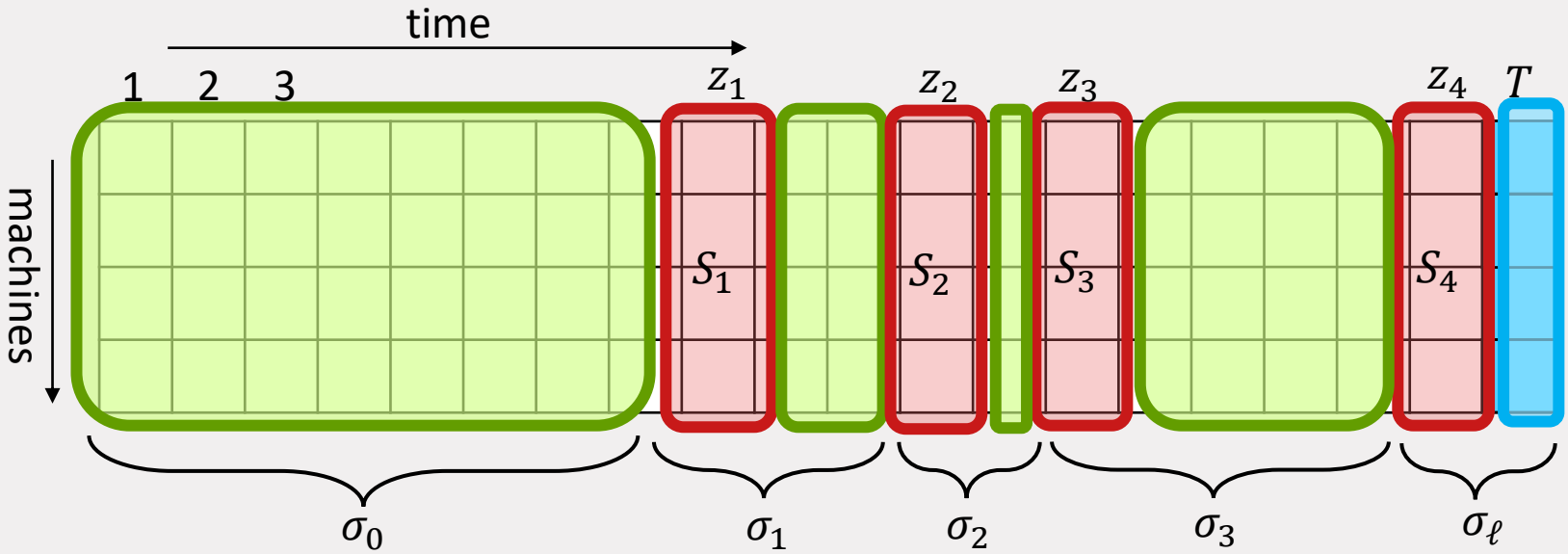

$$\sigma_i = \sigma(S_i, S_{i+1})$$

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Summary of Algorithm

Input: X a set of jobs, m

Output: minimal makespan of X

1. For each $S_1, S_2 \in \binom{X}{\leq m}$ do:
   a) Compute recursively makespan of $X \cap \sigma(S_1, S_2)$
2. Find a minimal combination of these subschedules
3. Return found schedule

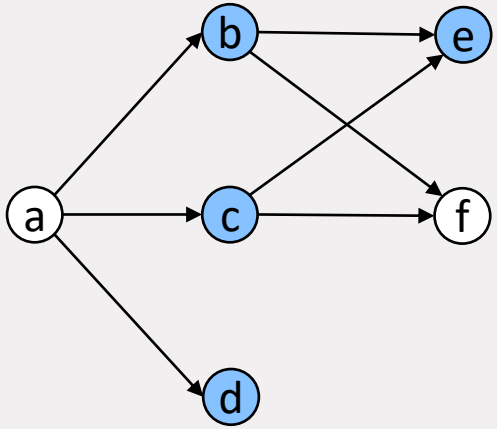
Simple analysis: $2^n \, poly(n)$ time…?

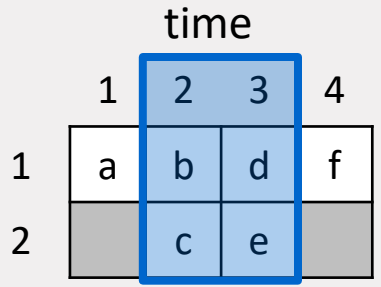
Only n^{2m} subproblems

Use of DP in $\approx n^{2m}$ time

Describing subschedules

Let σ' be a *subschedule*.





Describing subschedules

Let σ' be a *subschedule*.

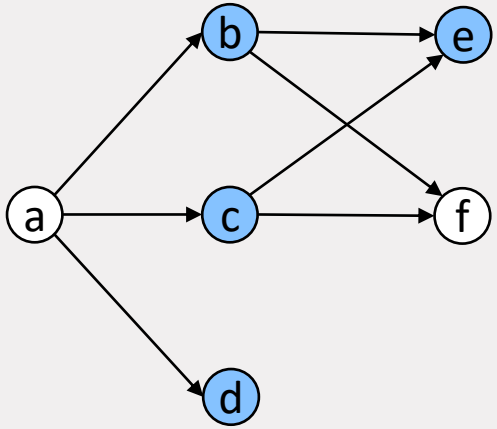
Jobs $V(\sigma')$ can be described as

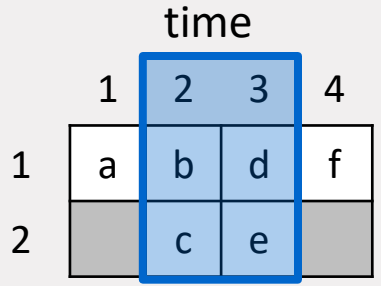
$$V(\sigma') = \text{succ}[A] \cap \text{pred}[B]$$

where

A = minimal elements = sources of σ'

B = maximal elements = sinks of σ'





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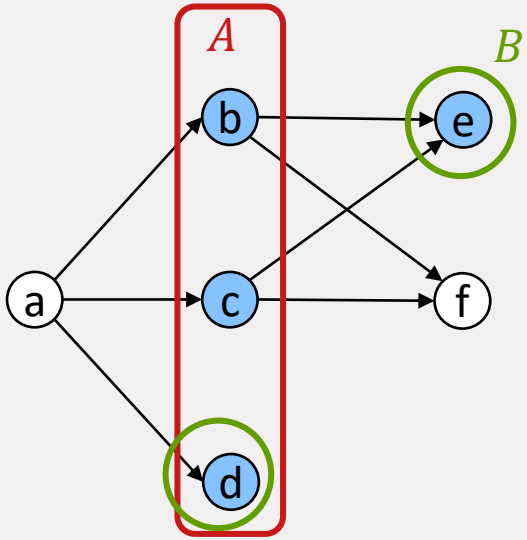

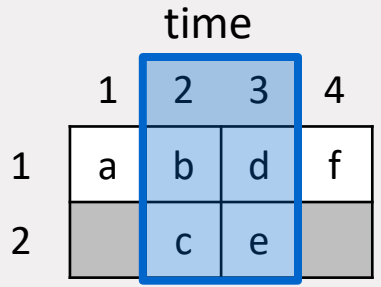

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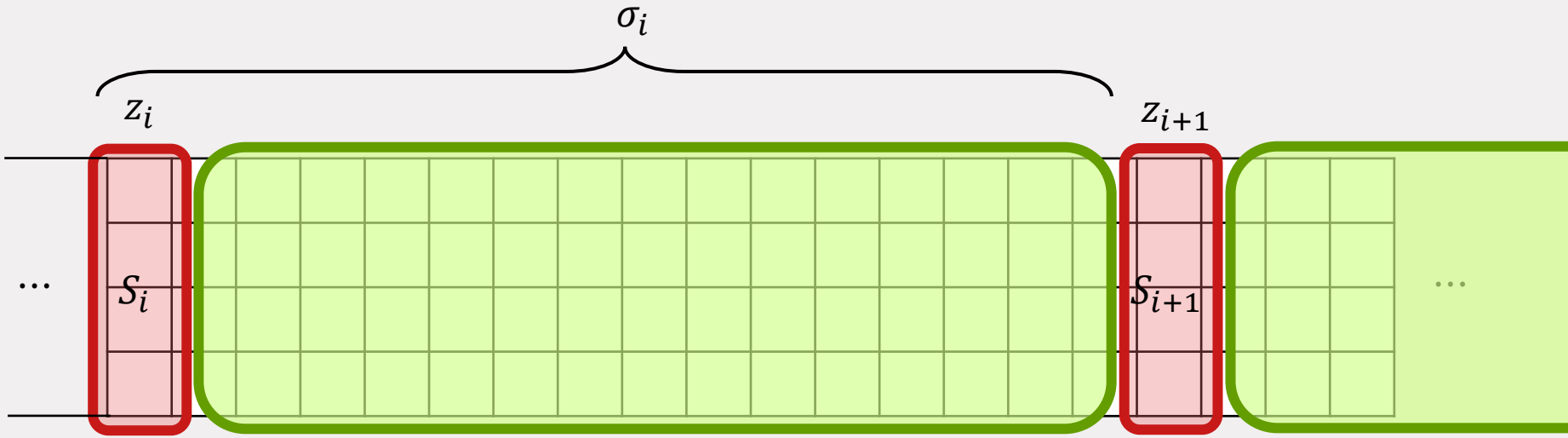
Sink-Adjusted Schedule



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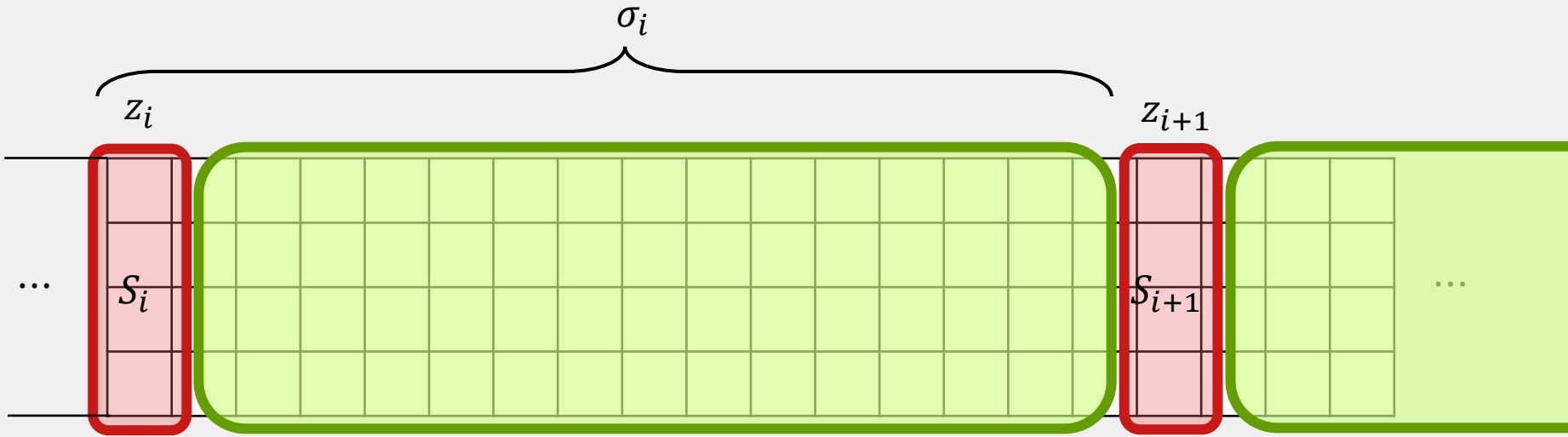

Lemma:

$$V(\sigma_i) \approx \mathrm{succ}[S_i] \setminus \left(\mathrm{succ}[S_{i+1}] \cup \mathrm{sinks}(\sigma)\right)$$

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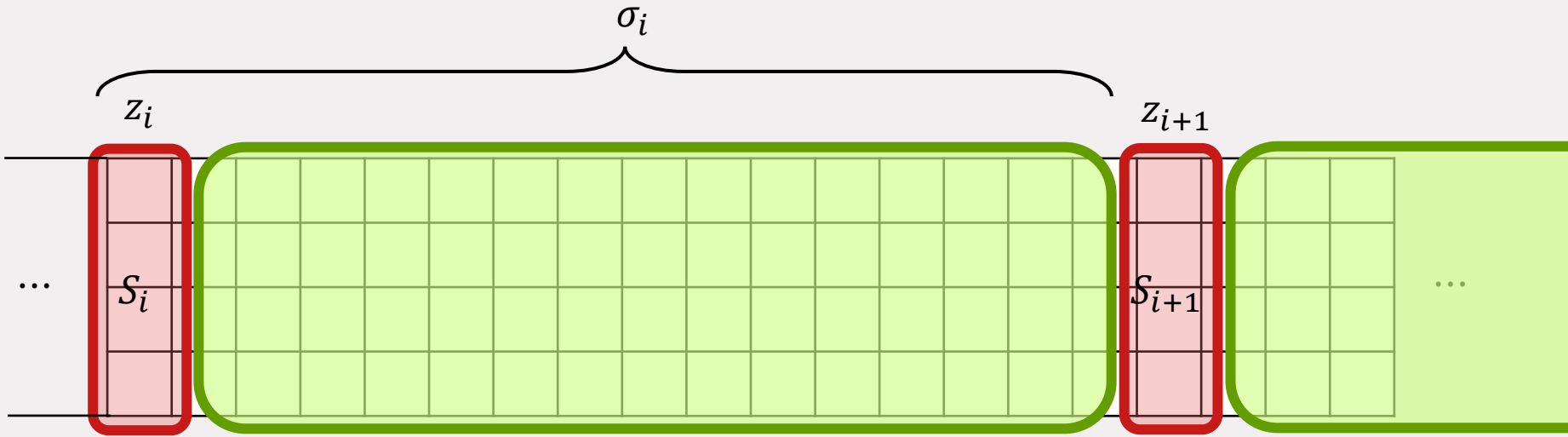

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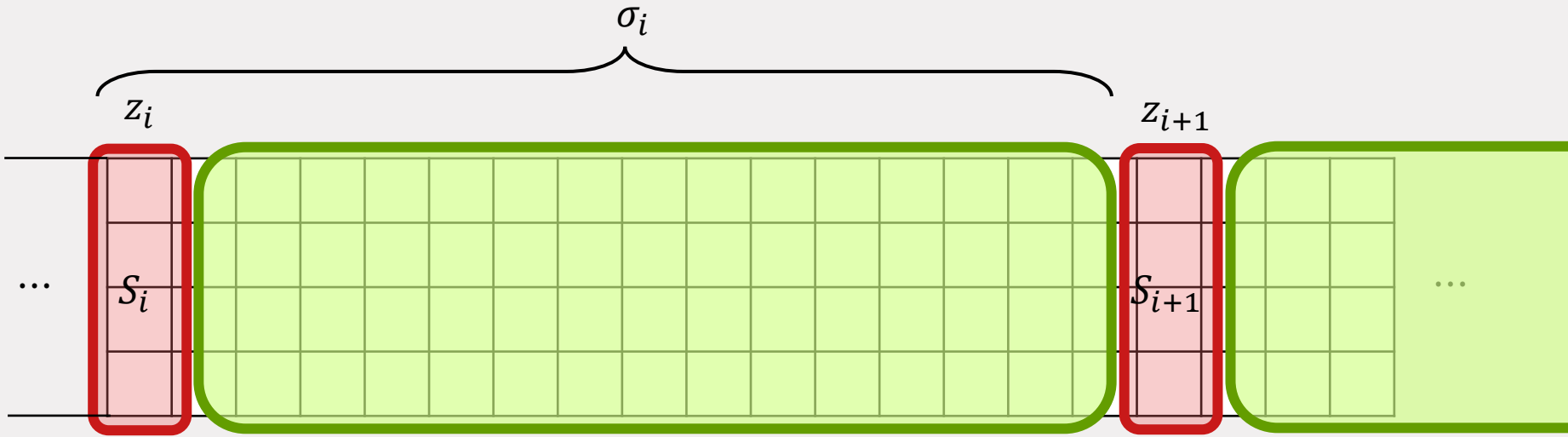
Sink-Adjusted Schedule



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n^m

win-win strategy

TU/e

Algorithm

Store subproblem results in lookup table:

$$T(A, B) \coloneqq \min \text{ makespan of } \text{succ}[A] \cap \text{pred}[B]$$

Use of lookup table, how many *different* subproblems?

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Remark: $|S_i| \leq m$!!

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⇒ at most $\binom{n}{m} \cdot \#\text{different } B$

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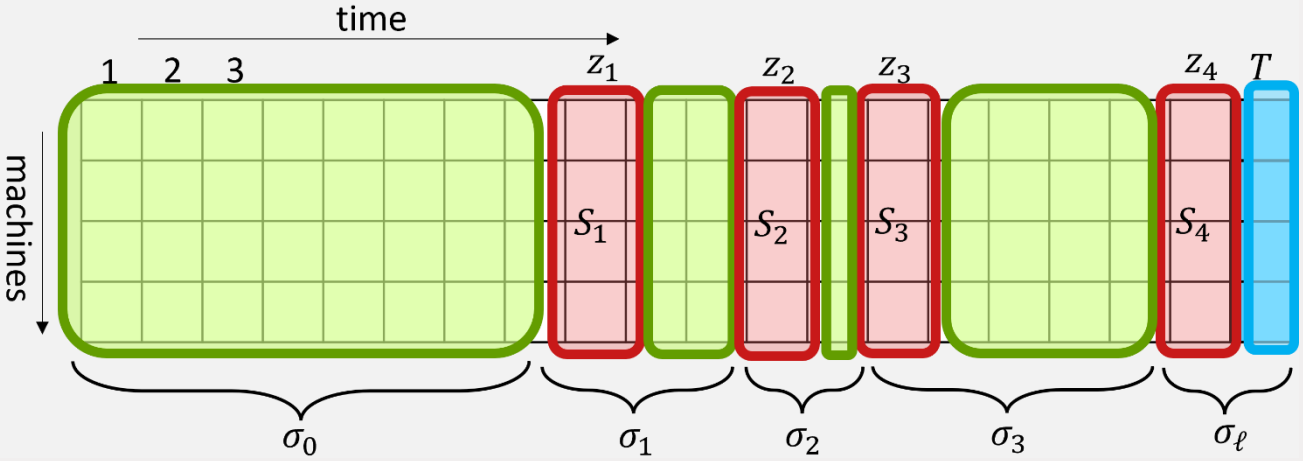

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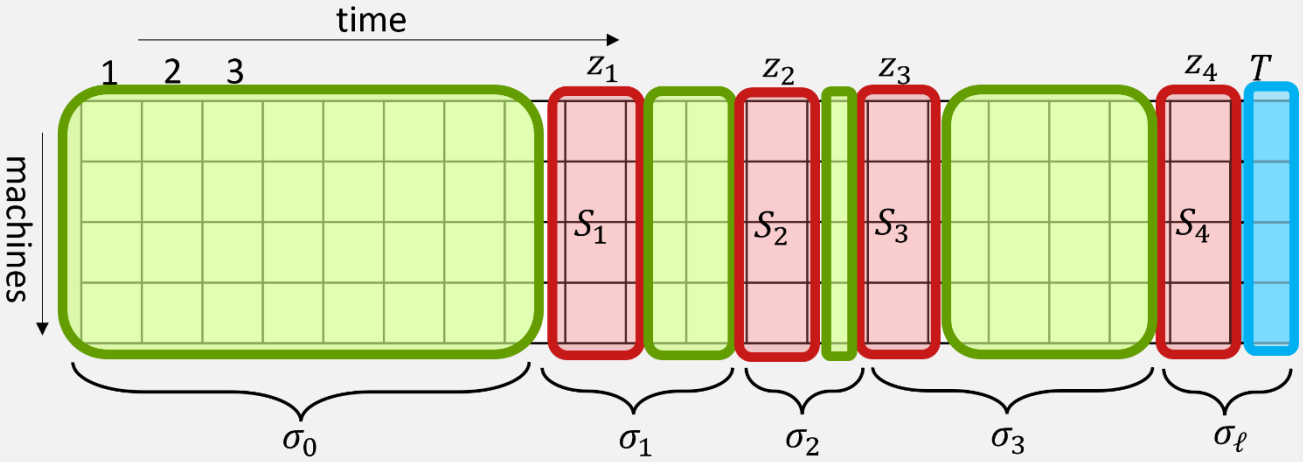

Each σ_i contains at most m sinks (and σ_ℓ is extremely easy)

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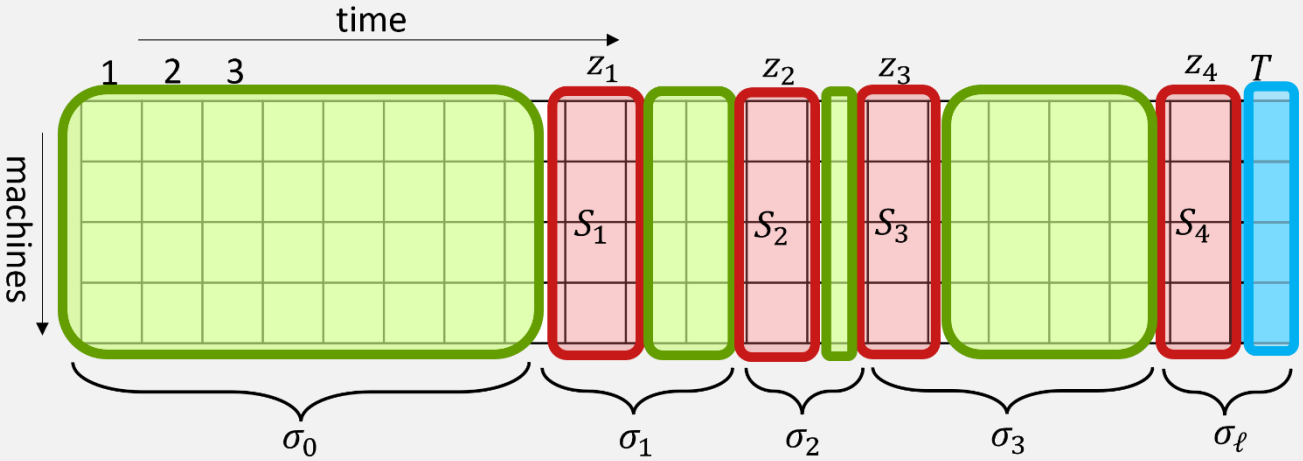

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$$|\sigma_i| \leq n - \sqrt{n} + m$$

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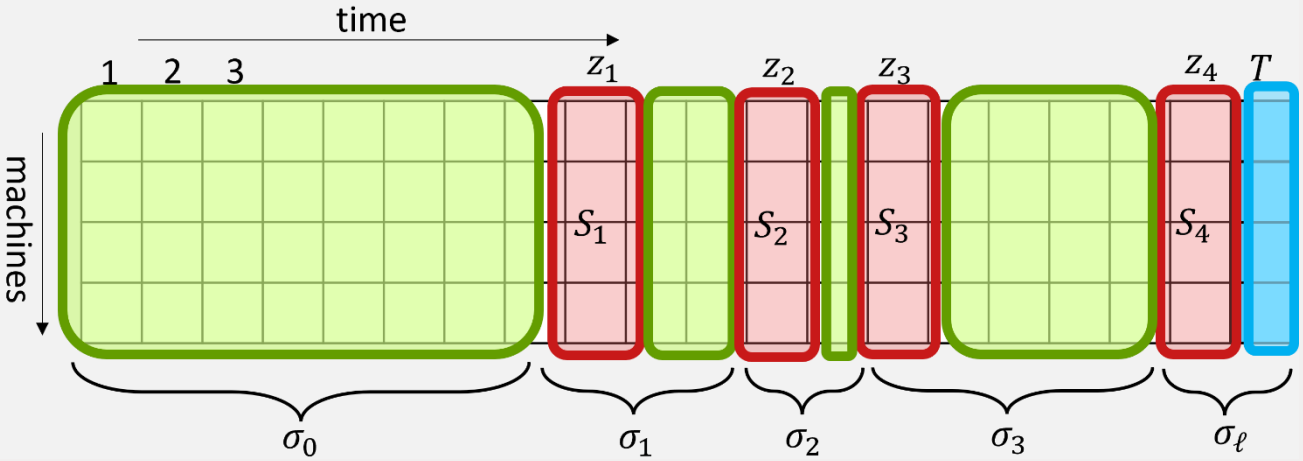

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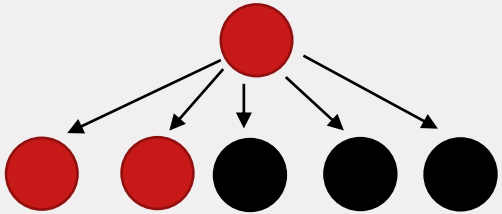
TU/e

Algorithm

$\leq n^{2m} = n^{O(1)}$ children



$= |B| \leq \sqrt{n}$

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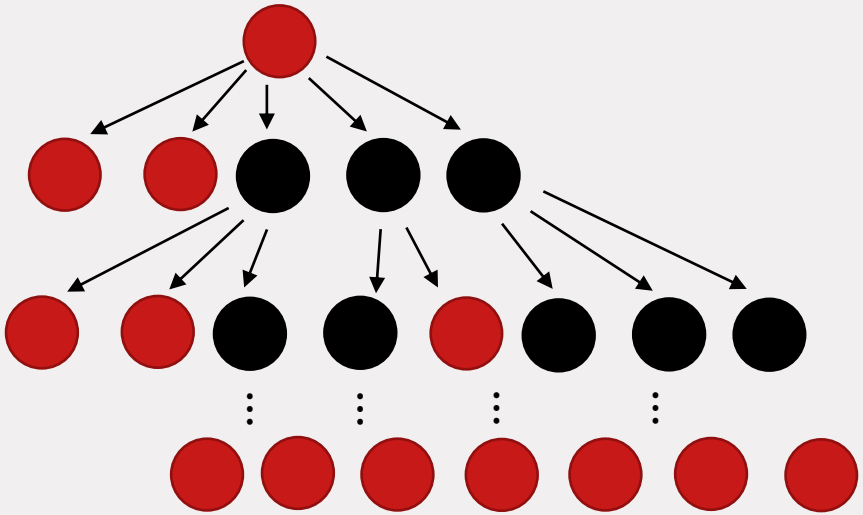
Algorithm

$\leq n^{2m} = n^{O(1)}$ children

Height $\leq \sqrt{n}$

$\leq \left(n^{O(1)}\right)^{\sqrt{n}} = 2^{O(\sqrt{n} \cdot \log n)}$ nodes in tree!



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TU/e

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Use of lookup table, how many *different* subproblems?

⇒ at most $2^{O(\sqrt{n} \cdot \log n)}$ trees with $2^{O(\sqrt{n} \cdot \log n)}$ nodes

⇒ $2^{O(\sqrt{n} \cdot \log n)}$ *different* subproblem!

Case $\lvert B \rvert \leq \sqrt{n}$	**Case** $\lvert B \rvert > \sqrt{n}$
⇒ only $\binom{n}{\sqrt{n}} = 2^{O(\sqrt{n} \cdot \log n)}$ different B's ✓	In next step: make $\sqrt{n}$ jobs progress! ✓

Summary of Algorithm

Input: A, B, m

Output: minimal makespan of $\mathrm{succ}[A] \cap \mathrm{pred}[B] = \mathrm{Jobs}$

1. For each $S_1, S_2 \in \binom{\mathrm{Jobs}}{\leq m}$ do:
   a) Compute recursively makespan of $\mathrm{Jobs} \cap (\mathrm{succ}[S_1] \setminus (\mathrm{succ}[S_2] \cup B)$
2. Find a minimal combination of these subschedules
3. Return found schedule

Conclusion

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Main result:

$Pm|prec, p_j = 1|C_{\max}$ in $2^{O(\sqrt{n} \cdot \log n)}$ time.

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- New decomposition of schedules
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$P3|prec, p_j = 1|C_{\max}$ in quasi-polynomial time?

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Thanks for your attention!